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THEOREM FOR SERIES IN THREE-PARAMETER MITTAG-LEFFLER FUNCTION

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Abstract

The new result presented here is a theorem involving series in the three-parameter Mittag-Leffler function. As a by-product, we recover some known results and discuss corollaries. As an application, we obtain the solution of a fractional differential equation associated with a *RLC* electrical circuit in a closed form, in terms of the two-parameter Mittag-Leffler function.

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1. Introduction

The applications of the Mittag-Leffler function and its extensions are discussed recently in a rapidly increasing number of papers, related to Fractional Calculus and fractional order differential and integral equations and systems, modeling various phenomena. Here, we mention only two results: one of them – by Magin and Ovadia [15], proposes modeling the cardiac tissue electrode interface using fractional calculus by means of a convenient three element electrical circuit. Camargo et al. [7] discuss the so-called telegraph equation in a fractional version whose solution is given in terms of a three-parameter Mittag-Leffler function and present also two new theorems involving the two- and three-parameter Mittag-Leffler functions.

On the other hand, in Hilfer's book [10], several papers can be seen as references, involving applications of the Mittag-Leffler functions associated with physical problems, in particular – problems involving fractional reaction-diffusion equations. Also, in Mainardi and Gorenflo [17], the authors present the solution of fractional diffusion equations in terms of the two parameters Mittag-Leffler function and/or in terms of the Wright functions, resp. [18].

In [6] it was discussed the calculation of several Green's functions associated with fractional differential equations by means of the juxtaposition of the integral transforms, all of them presented in terms of the Mittag-Leffler function. More recently, in [8] it was discussed the fractional Langevin equation, where the authors presented a solution in terms of the three-parameter Mittag-Leffler function. Also, the corresponding relaxation function is presented in terms of the convenient Mittag-Leffler functions. On the other hand, a new look at the fractional derivative by taking as starting point the Cauchy derivative formulation was presented by Ortigueira [21].

Finally, we can mention that other important and useful special functions that are related to the fractional calculus, as the Wright's function (called also Fox-Wright function), the Mainardi's function [16], the "vector" and "multi-index" Mittag-Leffler functions [12, 13, 14] have been studied as particular cases of the Fox's H-function [9], defined by means of the Mellin-Barnes integral [23, Vol. III], [19], [11].

In this paper, after a preliminaries of the Mittag-Leffler function, in Section 3, we present our main result – a theorem for series involving three-parameter Mittag-Leffler functions, and discuss some particular cases. In Section 4, we illustrate an application of our theorem, discussing a RLC electrical circuit and presenting the solution in a closed form. Concluding remarks close the paper.

2. Preliminaries on the Mittag-Leffler function

The Mittag-Leffler function E_α was introduced years ago in [20]. In the fifthies, Agarwal [1] introduced the two-parameter Mittag-Leffler function $E_{\alpha,\beta}$ as its generalization, usually referred to now as the (classical) Mittag-Leffler (M-L) function. An interesting review on these two functions can be found in Mainardi-Gorenflo [17], where the authors recall the main properties of the M-L functions and discuss some fractional differential equations whose solutions can be found in their terms. There are many other generalizations of the classical Mittag-Leffler function but here we are interested in only one of them, the so-called three-parameter Mittag-Leffler function, introduced by Prabhakar [22], and studied recently by Kilbas et al., see e.g. [11]:

$$E^{\rho}_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\mu k + \nu)} \frac{z^k}{k!}, \tag{1}$$

with $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(\rho) > 0$ and $z \in \mathbb{C}$. When the parameter $\rho = 1$, we get the two-parameter M-L function, as was introduced by Agarwal, and with $\rho = 1 = \nu$ we recover the M-L function as originally introduced by Mittag-Leffler. The case $\rho = 1 = \nu = \mu$ reduces to the exponential function. That is, the following relations are valid:

$$E^{1}_{\mu,\nu}(x) \equiv E_{\mu,\nu}(x)\ ,\ E^{1}_{\mu,1}(x) \equiv E_{\mu}(x)\ ,\ E^{1}_{1,1}(x) \equiv E_{1,1}(x) \equiv E_1(x) = \exp(x).$$

To close these preliminaries, let us remind the pair of the Laplace integral transform and the respective inverse, of the three-parameter M-L function (see for example in [11]), which will be used to discuss the *RLC* electrical circuit as illustration for applications of our results:

$$\mathfrak{L}[t^{\beta-1}E^{\rho}_{\alpha,\beta}(\pm\lambda t^{\alpha})] = \frac{s^{\alpha\rho-\beta}}{(s^{\alpha} \mp \lambda)^{\rho}}\ ,\ \mathfrak{L}^{-1}\left[\frac{s^{\alpha\rho-\beta}}{(s^{\alpha} \mp \lambda)^{\rho}}\right] = t^{\beta-1}E^{\rho}_{\alpha,\beta}(\pm\lambda t^{\alpha}),$$

with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) > 0$ and $\lambda \in \mathbb{C}$.

3. Some results for series in 3-parameter Mittag-Leffler function

In this section, first we derive a result involving the argument of the three-parameter M-L function in a product form related with an argument in a sum form. Then, we calculate explicitly this sum by presenting it in terms of two-parameter Mittag-Leffler functions. A corollary and some particular cases are discussed.

THEOREM 3.1. *Consider the parameters* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$, *and* $x, y \in \mathbb{C}$. *Then, for the three-parameter M-L function* $E^{\rho}_{\mu,\nu}(\cdot)$ *we have the relation*

$$\sum_{r=0}^{\infty}(x+y)^r E^{r+1}_{2\alpha,\alpha r+\beta}(-xy) = \sum_{k=0}^{\infty}(-xy)^k E^{k+1}_{\alpha,2\alpha k+\beta}(x+y)\,. \tag{2}$$

P r o o f. Substituting definition (1) in Eq.(2) and changing the sums, which are absolutely convergent, we have

$$\Lambda \equiv \sum_{k=0}^{\infty} \frac{(-xy)^k}{k!} \sum_{r=0}^{\infty} \frac{(r+1)_k}{\Gamma(2\alpha k + \alpha r + \beta)} (x+y)^r,$$

where $(\cdot)_k$ is the Pochhammer symbol.

Using the well-known relation

$$\frac{(r+1)_k}{k!} = \frac{\Gamma(r+k+1)}{k!\Gamma(r+1)} = \frac{\Gamma(k+r+1)}{r!\Gamma(k+1)} = \frac{(k+1)_r}{r!} \,,$$

we can write then:

$$\Lambda \equiv \sum_{k=0}^{\infty} (-xy)^k \sum_{r=0}^{\infty} \frac{(k+1)_r}{\Gamma(\alpha r + 2\alpha k + \beta)} \frac{(x+y)^r}{r!}.$$

Using again definition (1), we obtain

$$\Lambda \equiv \sum_{k=0}^{\infty} (-xy)^k E^{k+1}_{\alpha, 2\alpha k + \beta}(x+y)$$

which is exactly the second member of Eq.(2). The proof is complete. ■

This result implies that: we can change a sum involving a three-parameter Mittag-Leffler function whose argument is a product to the another three-parameter Mittag-Leffler function whose argument can be written as a sum. This can be called as a *particular sum rule*. When we calculate the sum, we really have this sum rule, as can be seen below.

THEOREM 3.2. *For the three-parameter Mittag-Leffler functions with* $Re(\alpha) > 0$ *and* $Re(\beta) > 0$ *and* $x, y \in \mathbb{C}$, $x \neq y$, *we have the following explicit representation of the series*

$$\sum_{k=0}^{\infty} (-xy)^k E^{k+1}_{\alpha, 2\alpha k + \beta}(x+y) = \frac{x E_{\alpha,\beta}(x) - y E_{\alpha,\beta}(y)}{x - y} \,, \tag{3}$$

in terms of the two-parameter Mittag-Leffler function $E_{\mu,\nu}(\cdot)$, *defined as in (1) with* $\rho = 1$.

P r o o f. To demonstrate this result, we first consider the binomial expansion

$$(x+y)^\ell = \sum_{n=0}^{\ell} \frac{\ell!}{n!(\ell-n)!} x^{\ell-n} y^n$$

and substitute the definition of three-parameter M-L function in the second member of Eq. (2) to obtain:

$$\Lambda = \sum_{k=0}^{\infty}(-xy)^k \sum_{\ell=0}^{\infty} \frac{(k+1)_\ell}{\Gamma(\alpha\ell+2\alpha k+\beta)\ell!} \sum_{n=0}^{\ell} \frac{\ell!}{n!(\ell-n)!} x^{\ell-n} y^n,$$

which can also be written as follows

$$\Lambda = \sum_{k=0}^{\infty} \frac{(-xy)^k}{k!} \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{\ell=0}^{\infty} \frac{\Gamma(k+\ell+n+1)}{\Gamma(\alpha\ell+\alpha n+2\alpha k+\beta)} \frac{x^\ell}{\ell!},$$

where in the last step, we have substituted $\ell \to \ell + n$.

Another convenient change of indices, $n \to n - k$ and $\ell \to \ell - k$ permit us to write

$$\Lambda = \sum_{n=0}^{\infty} y^n \sum_{\ell=0}^{\infty} \frac{x^\ell}{\Gamma(\alpha\ell+\alpha n+\beta)} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(\ell+n-k)!}{(n-k)!(\ell-k)!} \tag{4}$$

which for $n < k$ and $\ell < k$ is equal to zero.

Thus, the last sum in Eq. (4) is equal to one (see e.g. in [23, Vol. I], and then we obtain

$$\Lambda = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{x^\ell y^n}{\Gamma(\alpha\ell+\alpha n+\beta)}, \tag{5}$$

which can be put in the following form

$$\Lambda = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\Gamma(\alpha\ell+\beta)} \sum_{n=0}^{\ell} \left(\frac{y}{x}\right)^n.$$

Using the well-known expansion (e.g. [23, Vol. I])

$$\sum_{n=0}^{\ell} \left(\frac{y}{x}\right)^n = \frac{x^{\ell+1}-y^{\ell+1}}{x-y}, \quad x \neq y,$$

Eq. (5) can be written explicitly in the form

$$\Lambda = \frac{1}{x-y}\left[x\sum_{\ell=0}^{\infty} \frac{x^\ell}{\Gamma(\alpha\ell+\beta)} - y\sum_{\ell=0}^{\infty} \frac{y^\ell}{\Gamma(\alpha\ell+\beta)}\right].$$

To conclude the proof, we substitute in the last expression the two-parameter Mittag-Leffler function, obtained from (1) with $\rho = 1$. ■

Equation (3) is our main result. To discuss the case $y = x$, one must take the limit $y \to x$ and use the l'Hôpital rule, which is presented in the following corollary.

Corollary 3.3. *Let the parameters Re(α) > 0 and Re(β) > 0, and $x \in \mathbb{R}$. Then, the following representation holds:*

$$\sum_{k=0}^{\infty}(-x^2)^k E^{k+1}_{\alpha,2\alpha k+\beta}(2x) = E_{\alpha,\beta}(x) + x\frac{d}{dx}E_{\alpha,\beta}(x)\,, \qquad (6)$$

in terms of the two-parameter M-L function $E_{\mu,\nu}(\cdot)$.

P r o o f. To prove this result, it is sufficient to take the limit $y \to x$ in Eq. (3) and to use the l'Hôpital rule. ∎

On the other hand, taking the limit $y \to -x$ in Eq. (3), we get

$$\sum_{k=0}^{\infty} x^{2k} E^{k+1}_{\alpha,2\alpha k+\beta}(0) = \frac{1}{2}\left[E_{\alpha,\beta}(x) + E_{\alpha,\beta}(-x)\right]\,,$$

which can be written as follows:

$$\sum_{k=0}^{\infty} x^{2k} E^{k+1}_{\alpha,2\alpha k+\beta}(0) = E_{2\alpha,\beta}(x^2)\,.$$

This is an identity for the two-parameter Mittag-Leffler function.

Particular cases

Here we discuss the particular case $\alpha = 1$. Putting $\alpha = 1$ in Eq. (3), we can write

$$\sum_{k=0}^{\infty}(-xy)^k\, E^{k+1}_{1,2k+\beta}(x+y) = \frac{xE_{1,\beta}(x) - yE_{1,\beta}(y)}{x-y}.$$

Using the relation

$$\Gamma(\mu)E^{\rho}_{1,\mu}(x) = {}_1F_1(\rho;\mu;x)\,,$$

where ${}_1F_1(\rho;\mu;x)$ is the confluent hypergeometric function, we get

$$\sum_{k=0}^{\infty}(-xy)^k\,\frac{{}_1F_1(k+1;2k+\beta;x+y)}{\Gamma(2k+\beta)} = \frac{xE_{1,\beta}(x) - yE_{1,\beta}(y)}{x-y}\,.$$

In the subcase $\beta = 1$, this gives

$$\sum_{k=0}^{\infty}\frac{(-xy)^k}{(2k)!}\,{}_1F_1(k+1;2k+1;x+y) = \frac{x\,\mathrm{e}^x - y\,\mathrm{e}^y}{x-y}\,, \quad \text{for } y \neq x\,.$$

Using the l'Hôpital rule to calculate the limit $y \to x$, we obtain

$$\sum_{k=0}^{\infty}\frac{(-x^2)^k}{(2k)!}\,{}_1F_1(k+1;2k+1;2x) = (1+x)\,\mathrm{e}^x.$$

On the other hand, the subcase $\beta = 2$ furnishes

$$\sum_{k=0}^{\infty}(-xy)^k \frac{{}_1F_1(k+1;2k+2;x+y)}{\Gamma(2k+2)} = \frac{\mathrm{e}^x - \mathrm{e}^y}{x-y}\,,$$

and taking the limit $y \to x$, we have

$$\sum_{k=0}^{\infty}(-x^2)^k \frac{{}_1F_1(k+1;2k+2;2x)}{\Gamma(2k+2)} = \mathrm{e}^x.$$

These two latter results have been obtained in a recent paper [7].

Finally, setting $\beta = 2\alpha$ in Eq. (3), we recover the results associated with the telegraph equation which have been discussed also in [7].

4. Illustrating application

To illustrate our results, as an application we discuss a particular RLC electrical circuit, as below. In this electrical circuit, the capacitor and the inductor are connected in parallel and this set is connected with a resistor in series. A source is considered of the Heaviside type. We remember that a similar electrical circuit was recently studied by Magin and Ovadia [15], but in that paper the authors substitute the inductor by an impedance.

4.1. RLC Electrical circuit

In this section we present a RLC electrical circuit with a capacitor and an inductor are connected in parallel and this set is connected in series with a resistor, and a voltage. The capacitance, C, the inductance, L and the resistance, R, are considered positive constants and $\theta(t)$ is the Heaviside function.

The constitutive equations associated with the three-elements of the RLC electrical circuit are: the voltage drop $U_L(t) = L\,\dfrac{\mathrm{d}}{\mathrm{d}t}I(t)$, across an inductor; the voltage drop $U_R(t) = R\,I(t)$, across a resistor; the voltage drop $U_C(t) = \frac{1}{C}\int^t I(\xi)\mathrm{d}\xi$, across a capacitor, and where $I(t)$ is the current.

Using the Kirchhoff's voltage law and the constitutive equations associated with the three elements, we can write the non-homogeneous second order ordinary differential equation

$$RC\frac{\mathrm{d}^2}{\mathrm{d}t^2}U_C(t) + \frac{\mathrm{d}}{\mathrm{d}t}U_C(t) + \frac{R}{L}U_C(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t)\,, \tag{7}$$

where $U_C(t)$ is the voltage on the capacitor which is the same on the inductor, as we can see in Figure 1, because they are connected in parallel.

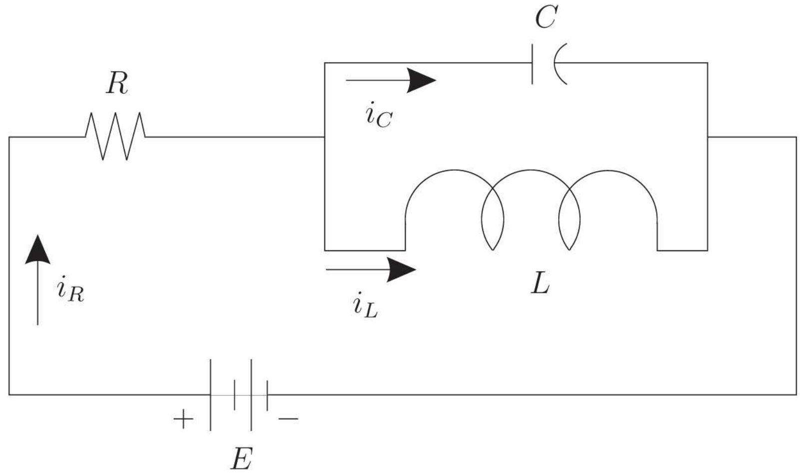

Fig.1: Three-Element Electrical Circuit

On the other hand, we obtain another non-homogeneous second order ordinary differential equation associated with the current on the inductor,

$$RLC\frac{\mathrm{d}^2}{\mathrm{d}t^2}i_L(t)+L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t)+Ri_L(t)=\theta(t). \tag{8}$$

Again, using the constitutive equation for the inductor, these two non-homogeneous second order ordinary differential equations can be led to the correspondent integro-differential equations,

$$R\frac{\mathrm{d}}{\mathrm{d}t}i_C(t)+\frac{1}{C}i_C(t)+\frac{R}{LC}\int_0^t i_C(\xi)\,\mathrm{d}\xi=\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \tag{9}$$

and

$$RC\frac{\mathrm{d}}{\mathrm{d}t}U_L(t)+U_L(t)+\frac{R}{L}\int_0^t U_L(\xi)\,\mathrm{d}\xi=\theta(t)\,, \tag{10}$$

respectively. We note that, these integro-differential equations have the same form. Here we consider only the first one. The classical methodology to discuss this integro-differential equation is the Laplace transform. To this end, we consider the initial condition $i_C(0)=0$ and the solution can be found in terms of an exponential function, see e.g. [24].

4.2. Fractional integro-differential equation

In this subsection we discuss the fractional version of Eq. (9), i.e., a fractional integro-differential equation associated with the current on the capacitor,

$$R\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}i_C(t) + \frac{1}{C}i_C(t) + \frac{R/LC}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\, i_C(\xi)\,\mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \tag{11}$$

with $0 < \alpha \leq 1$, and the fractional derivative is taken in the Caputo sense, where $\theta(t)$ is the Heaviside function. In this case, one can thought that $i_C(t)$ can be interpreted as a Green's function because the second member is a delta function. We also consider $i_C(0) = 0$, i.e., the initial current on the capacitor is zero. We note that this equation is a possible generalization of the classical integro-differential equation associated with the RLC electrical circuit, because for $\alpha = 1$ we recover the results obtained in Subsection 4.1. This replacement can be useful in discussing the corresponding numerical problem, for a particular value of the parameter, because the solution is presented in terms of a closed expression.

To solve this fractional integro-differential equation, we introduce the Laplace integral transform, defined by

$$\mathfrak{L}[i_C(t)] \equiv F(s) = \int_0^\infty \mathrm{e}^{-st} i_C(t)\,\mathrm{d}t$$

with $\mathrm{Re}(s) > 0$, and we obtain the following algebraic equation

$$Rs^\alpha F(s) + \frac{F(s)}{C} + \frac{R/LC}{s^\alpha}F(s) = 1\,,$$

whose solution is given by

$$F(s) = \frac{1}{R}\frac{s^\alpha}{s^{2\alpha} + as^\alpha + b}\,,$$

where we have introduced the positive parameters $a \equiv 1/RC$ and $b \equiv 1/LC$.

To recover the solution of the fractional integro-differential equation, we proceed with the inverse Laplace transform

$$i_C(t) = \frac{1}{R}\mathfrak{L}^{-1}\left[\frac{s^\alpha}{s^{2\alpha} + as^\alpha + b}\right].$$

Using the relation (from [3])

$$\mathfrak{L}^{-1}\left[\frac{s^{\rho-1}}{s^\alpha + As^\beta + B}\right] = t^{\alpha-\rho}\sum_{r=0}^\infty (-A)^r t^{(\alpha-\beta)r} E^{r+1}_{\alpha,\alpha+1-\rho+(\alpha-\beta)r}(-Bt^\alpha)$$

valid for $|A\,s^\beta/(s^\alpha + B)| < 1$ and $\alpha \geq \beta$, we can write

$$i_C(t) = \frac{t^{\alpha-1}}{R}\sum_{r=0}^\infty (-a)^r t^{\alpha r} E^{r+1}_{2\alpha,\alpha+\alpha r}(-bt^{2\alpha})\theta(t)\,, \tag{12}$$

where $E^{\rho}_{\mu,\nu}(\cdot)$ is the three-parameter Mittag-Leffler function (1), and $\theta(t)$ is the Heaviside function.

To evaluate explicitly this sum, we use Theorem 3.2 and get

$$i_C(t) = \frac{t^{\alpha-1}}{R}\frac{\mu E_{\alpha,\alpha}(\mu\, t^{\alpha}) - \nu E_{\alpha,\alpha}(\nu\, t^{\alpha})}{\mu-\nu}\theta(t)\,, \tag{13}$$

where μ and ν are the roots of the algebraic system $\mu+\nu = -1/RC$ and $\mu\nu \equiv 1/LC$.

5. Concluding remarks

In this paper we propose new results for series in three-parameter Mittag-Leffler functions, presenting them explicitly in terms of the two-paramater (classic) Mittag-Leffler functions. To illustrate the possible applications of our results, we obtain a closed form to the solution of the fractional integro-differential equation associated with a particular RLC electrical circuit, in terms of the two-parameter Mittag-Leffler function $E_{\alpha,\beta}$.

Our main result is interesting with respect to simplifying several other results, for example, as one can see in [5] where we discussed the fractional telegraph equation, and in [4], where the anomalous diffusion was presented. The results in both papers are given in terms of the three-parameter Mittag-Leffler function. A natural continuation of this paper is to discuss the gamma distribution whose density function is known. This study is in progress [2].

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References

[1] R.P. Agarwal, A propos d'une note de M. Pierre Humbert. *C.R. Acad. Sci. Paris*, **236** (1953), 2031-2032.

[2] R. Charnet and E. Capelas de Oliveira, On the product of two gamma distributions and the Mittag-Leffler function. Submitted.

[3] R. Figueiredo Camargo, *Fractional Calculus and Applications* (in Portuguese). Doctoral Thesis, Unicamp, Campinas, 2009.

[4] R. Figueiredo Camargo, E. Capelas de Oliveira and J. Vaz Jr., On the anomalous diffusion and the fractional generalized Langevin equation. *J. Math. Physics* **50**, (2009) 123518 (publ. online Dec. 2009).

[5] R. Figueiredo Camargo, E. Capelas de Oliveira and J. Vaz Jr., On the generalized Mittag-Leffler function and its application in a fractional telegraph equation. Submitted for publication, 2009.

[6] R. Figueiredo Camargo, R. Charnet and E. Capelas de Oliveira, On some fractional Green's functions. *J. Math. Phys.* **50** (2009), 043514.

[7] R. Figueiredo Camargo, A.O. Chiacchio and E. Capelas de Oliveira, Differentiation to fractional orders and the telegraph equation. *J. Math. Phys.* **49** (2008), 033505.

[8] R. Figueiredo Camargo, A.O. Chiacchio, R. Charnet and E. Capelas de Oliveira, Solution of the fractional Langevin equation and the Mittag-Leffler functions. *J. Math. Phys.* **50** (2009) 063507.

[9] Ch. Fox, The G- and H- functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, **98** (1961), 395-429.

[10] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*. World Scientific, Singapore (2000).

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

[12] V. Kiryakova, Multiindex Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type integral transforms. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 445-462.

[13] V. Kiryakova, The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus. *Computers and Math. with Appl.* **59**, No 5 (2010), 1885-1895 (doi:10.1016/j.camwa.2009.08.025).

[14] Yu. Luchko, Operational method in fractional calculus. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 463-488.

[15] R.L. Magin and M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus. *J. Vibration and Control* **14** (2008), 1431-1442.

[16] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation. In: S. Rionero and T. Ruggeri (Eds.), *Waves and Stability in Continuous Media*, World Scientific, Singapore (1994), 246-251.

[17] F. Mainardi and R. Gorenflo, On Mittag-Leffler functions in fractional evolution processes. *J. Comp. Appl. Math.* **118** (2000), 283-299.

[18] F. Mainardi and G. Pagnini, The Wright functions as solutions of the time-fractional diffusion equations. *Appl. Math. Comp.* **141** (2003), 51-62.

[19] F. Mainardi and G. Pagnini, Salvatore Pincherle: The pioneer of the Mellin-Barnes integrals. *J. Compt. Appl. Math.* **153** (2003), 331-342.

[20] G.M. Mittag-Leffer, Sur la nouvelle fonction $E_\alpha(x)$. *C.R. Acad. Sci. Paris (Ser.II)* **137** (1903), 554-558.

[21] M.D. Ortigueira, A coherent approach to non-integer order derivatives. *Signal Processing* **86** (2006), 2505-2515.

[22] T.R. Prabhakar, A singular integral equation with generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19** (1971), 7-25.

[23] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integral and Series*, Vols. I – III. Gordon and Breach, New York (1986).

[24] A.L. Soubhia and E. Capelas de Oliveira, Electrical circuit and the fractional calculus (in Portuguese). Presented at: *XXXII CNMAC, Cuiabá, Brazil'2009*, 08-11 September, 2009.

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