ON A CONVEXITY PRESERVING INTEGRAL OPERATOR

Gheorghe Oros * and Georgia Irina Oros **

Abstract

Let $c$ be a complex number, with $Re \, c > 0$ and let $g$ be an analytic function in the unit disc, $U = \{z \in \mathbb{C}; |z| < 1\}$ with $g(0) = 0$, $g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$. In this paper we determine conditions an analytic function $g$ needs to satisfy in order that the function $F$ given by (1) be convex.

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1. Introduction and preliminaries

Let $U$ be the unit disc of the complex plane:

$$U = \{z \in \mathbb{C}; |z| < 1\}.$$ 

Let $\mathcal{H}(U)$ denote the class of analytic functions in $U$. Also, let

$$A_n = \{f \in \mathcal{H}(U); f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}$$

with $A_1 = A$, 

$$K = \left\{ f \in A, \ Re \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$$
denote the class of normalized convex functions in $U$,

$$C = \left\{ f \in A : \exists \varphi \in K, \ Re \frac{f'(z)}{\varphi'(z)} > 0, \ z \in U \right\}$$

denote the class of close-to-convex functions.

In order to prove our original result, we use the following lemma:

**Lemma A.** ([9]) If $P$ is an analytic function in $U$, with $ReP(0) > 0$ and if $P$ satisfies

$$Re \left[ P(z) + \frac{zP'(z)}{P(z)} \right] > 0, \ z \in U,$$

then $ReP(z) > 0, z \in U$.

Let $c$ be a complex number, with $Re c > 0$ and $g \in \mathcal{H}(U)$, with $g(0) = 0$, $g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$. Consider the integral operator $I : \mathcal{H}(U) \to \mathcal{H}(U)$ defined by $F = I(f)$, where

$$F(z) = \frac{1}{|g(z)|^c} \int_0^z f(w)g(w)^c-1g'(w)dw, \ z \in U, f \in \mathcal{H}(U). \quad (1)$$

It is well-known that in the particular case $g(z) = z$ and $c = 1$, Libera [3] proved that the operator $I$ preserves the starlikeness, the convexity and the close-to-convexity. This remarkable result was extended by many other authors (see, for example [1], [2], [4], [5], [6], [7], [12]-[14]).

For $c$ a complex number, with $Re c > 0$, and $g(z) = z$ similar results were obtained in [10] and [11] for the Bernardi integral operator.

In the case $c = 1$, sufficient conditions on the function $g$ such that $I$ is a convexity-preserving operator were given in [8].

In [9] the author shows that if $g$ satisfies the condition

$$Re[czg'(z)/g(z)] > 0$$

in $U$ and if the integral operator $I$ preserves the convexity, then $I$ also preserves the close-to-convexity.

In this paper we show that if $g$ satisfies the conditions

$$Re \frac{czg'(z)}{g(z)} > 0$$

and

$$Re \left[ \frac{zg''(z)}{g'(z)} + 1 \right] > Re(c + 1) \frac{zg'(z)}{g(z)}$$

in $U$ and if the integral operator $I$ preserves the close-to-convexity, then $I$ also preserves the convexity.
2. Main result

**Theorem 1.** Let $I$ be the integral operator defined by (1) and suppose that

(i) $\text{Re} \frac{czg'(z)}{g(z)} > 0$, $z \in U$, $\text{Re} \, c > 0$,

(ii) $\text{Re} \left[ \frac{zg''(z)}{g(z)} + 1 \right] > \text{Re} \frac{(c+1)zg'(z)}{g(z)}$, $z \in U$,

(iii) $I(C) \subset C$

then

$I(K) \subset K$.

**Proof.** If we let 

$$G(z) = \frac{g(z)}{zg'(z)}, \quad z \in U,$$

then the condition (i) implies $G \in H(U)$ and $G(z) \neq 0$ in $U$.

From (1), we obtain

$$zf'(z)G(z) + cf(z) = f(z), \quad z \in U$$

and

$$zf''(z)G(z) + [zG'(z) + G(z) + c]f'(z) = f'(z), \quad z \in U.$$

Let $f \in C$. Then there exists $\varphi \in K$, such that

$$\text{Re} \frac{f'(z)}{\varphi'(z)} > 0, \quad z \in U.$$

If we denote $\phi = I(\varphi)$, then

$$\phi(z) = \frac{1}{[g(z)]^c} \int_0^z \varphi(w)[g(w)]^{c-1}g'(w)dw, \quad \text{Re} \, c > 0. \quad (2)$$

Next we prove that $\phi \in K$.

Differentiating (2), we obtain

$$zf''(z)G(z) + [zG'(z) + G(z) + c]\phi'(z) = \varphi'(z), \quad z \in U$$

which is equivalent to

$$G(z)\frac{\phi'(z)}{\phi'(z)} \left[ \frac{zf''(z)}{\phi'(z)} + \frac{zG'(z)}{G(z)} + 1 + \frac{c}{G(z)} \right] = \varphi'(z). \quad (3)$$

If we let

$$P(z) = \frac{zf''(z)}{\phi'(z)} + \frac{zG'(z)}{G(z)} + 1 + \frac{c}{G(z)}, \quad z \in U, \quad (4)$$
then (3) becomes
\[ G(z) \cdot \phi'(z) \cdot P(z) = \varphi'(z), \quad z \in U. \] (5)

Differentiating (5), we obtain
\[ \frac{zG'(z)}{G(z)} + \frac{z\phi''(z)}{\phi'(z)} + \frac{zP'(z)}{P(z)} = \frac{z\varphi''(z)}{\varphi'(z)}, \quad z \in U \]
which is equivalent to
\[ \frac{zG'(z)}{G(z)} + \frac{z\phi''(z)}{\phi'(z)} + 1 + \frac{zP'(z)}{P(z)} = \frac{z\varphi''(z)}{\varphi'(z)} + 1 + \frac{c}{G(z)}, \quad z \in U. \] (6)

Using (4) in (6), we obtain
\[ P(z) + \frac{zP'(z)}{P(z)} = \frac{z\varphi''(z)}{\varphi'(z)} + 1 + \frac{c}{G(z)}, \quad z \in U. \] (7)

Using condition (i) from hypothesis and since \( \varphi \) is convex, we have
\[ \Re \left[ P(z) + \frac{zP'(z)}{P(z)} \right] = \Re \left[ \frac{z\varphi''(z)}{\varphi'(z)} + 1 + \frac{c}{g(z)} \right] > 0, \quad z \in U, \]
i.e.
\[ \Re \left[ P(z) + \frac{zP'(z)}{P(z)} \right] > 0, \quad z \in U. \] (8)

Letting \( z = 0 \) in (8), we deduce
\[ \Re P(0) > 0, \quad z \in U. \]

We have now the conditions from the hypothesis of Lemma A and applying it we obtain
\[ \Re P(z) > 0, \quad z \in U. \]

From \( G(z) = \frac{g(z)}{zg'(z)} \), we have
\[ \frac{zG'(z)}{G(z)} = \frac{zg'(z)}{g(z)} - \frac{zg''(z)}{g'(z)} - 1, \quad z \in U. \]

Using (4) and the condition \( \Re P(z) > 0, z \in U \) we obtain
\[ \Re \left[ \frac{z\phi''(z)}{\phi'(z)} + 1 + \frac{zG'(z)}{G(z)} + \frac{c}{G(z)} \right] > 0, \]
and using (ii), we obtain

\[ \text{Re} \left[ \frac{z\phi''(z)}{\phi'(z)} + 1 \right] > \text{Re} \left[ \frac{zg''(z)}{g'(z)} + 1 - \frac{(c + 1)zg'(z)}{g(z)} \right] > 0, \quad z \in U, \]

i.e.

\[ \text{Re} \left[ \frac{z\phi''(z)}{\phi'(z)} + 1 \right] > 0, \quad z \in U \]

which shows that \( \phi \in K \).

References


*Department of Mathematics, University of Oradea*

*Str. Universității, No.1*

*410087 Oradea – ROMANIA*

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