IN MEMORIAM C. S. MEIJER *

B.L.J. Braaksma

Editorial Note

This renewal publication of the article dedicated to the memory of the Dutch mathematician Prof. C.S. Meijer, has been inspired by my recent visit to Holland.

Incidentally, the year 1974 of his decease coincides with the time when, as a beginner in the field of integral transforms and special functions, I "stuck on" ones of his main contributions, Mejler's G-function and the Meijer integral transform. At that time, I had to "discover" the G-function as the "unknown" special function in the kernel of a Laplace type integral transform, introduced by a Bulgarian mathematician N. Obrechkoff as a generalization of the Meijer transform. Later, my studies on that transform and on the related "Bessel type" operators of arbitrary integer order have become an endless source for further G-function's applications, and completely new achievements, as the generalized operators of integration and differentiation of fractional multiorder. Details can be seen in the recently published survey, V. Kiryakova [68], see same issue, p. 244.

Another good provocation, to pay tribute to the achievements of C.S. Meijer, is the increasing attention that many authors in the fields of our "FCAA" Journal have recently paid to the history, development and applications of the so-called Mellin-Barnes type integrals, giving rise to the most useful definition of the G-function. Nowadays, this generalized hypergeometric function happens to provide: solutions to wide classes of ordinary

---

* Special thanks are due to Editors of *"Nieuw Archief voor Wiskunde (3)"*, where this article was originally published in Vol. XXIII (1975), pp. 95-104, for the permission of its reprinting in *"FCAA"*

differential equations of arbitrary order with variable coefficients and to Volterra type integral equations; kernel-functions for generalized integral transforms (referred to as G-transforms) and for variety of linear integral and integro-differential operators of applied analysis; as well as unification scheme for the other special functions.

I use this opportunity also to thank Profs. B.L.J. Braaksma, J. Boersma and J. de Graaf, for providing me copies of very rare literature on C.S. Meijer's and his collaborators' contributions, as well as for arranging my visit, talk and thrilling meetings at the Groningen University, the place where Prof. Meijer had worked.

Virginia Kiryakova, Managing Editor of "FCAA"

Prof. Dr. C.S. Meijer

*

Cornelis Simon Meijer was born August 17, 1904 in Pieterburen. He studied mathematics at the university of Groningen from 1924 until 1929 under the direction of professors Van der Corput and Van der Waerden. In 1933 he presented his doctor's thesis: *Asymptotische Entwicklungen Besselscher, Hankelscher und verwandter Funktionen, Bestimmung von numerischen oberen Schranken für das Restglied mittels der Methode der Sattelpunkte*, with prof. Van der Corput as promotor. He was professor of mathematics at the university of Groningen from 1946 until his retirement in 1972. He died on April 12, 1974 at the age of 69.

His scientific work consists of asymptopic expansions with error estimates and integral representations for special functions of mathematical physics, in particular Bessel and Whittaker functions, the generalizations of the Laplace transform which carry his name, and the "Meijer G-function".

His first papers [1-3] and his thesis [4] are concerned with asymptotic expansions of Bessel, Hankel and related functions. A special feature is the precise numerical bounds for the remainder term in these expansions. Earlier more restrictive results were given by Schläfli, Hankel, Weber, Watson and Van Veen. Meijer attacked these problems anew by means of the saddle point method of Debije. At that time most applications of this method did not involve numerical bounds for remainder terms with exception of the work of Van Veen. Meijer succeeded in obtaining precise numerical bounds valid for larger regions of the variables and parameters than previous writers by using a version of Lagrange's theorem on the expansion of implicitly given function with a remainder term. His method runs as follows.

Suppose the behaviour of

$$I(\lambda) = \int_a^b e^{\lambda\phi(z)}\psi(z)dz$$

is to be investigated as $\lambda \to \infty$. Suppose ψ and ϕ are analytic functions, $\phi'(a) = 0$, $\phi''(a) \neq 0$, $\phi(a) - \phi(z) > 0$ if $a < z \leq b$. Then one substitutes $\xi = \phi(a) - \phi(z)$, so that

$$I(\lambda) = e^{\lambda\phi(a)} \int_0^\beta e^{-\lambda\xi}\psi(z)\frac{dz}{d\xi}d\xi.$$

Now one may expand $\psi(z)\dfrac{dz}{d\xi}$ in a power series in ξ according to Lagrange. However, MEIJER writes $\psi(z)\dfrac{dz}{d\xi}$ as a partial sum of this series plus a remainder term in the form of an integral which may be suitably estimated. Thus he obtains sharp bounds for the error term in the asymptotic expansion of $I(\lambda)$. His results on asymptotic expansions with error estimates of Bessel and Hankel functions were for a long time the most complete in this field and are of importance for numerical purposes. Recent results in this area have been given by F.W.J. OLVER using asymptotic methods of differential equations.

In his thesis MEIJER exploited suitable integral representations of Bessel and Hankel functions in order to obtain asymptotic expansions. Integral representations of special functions of mathematical physics was his main field of interest in subsequent years. During the period 1934 - 1941 MEIJER and A. ERDÉLYI developed the theory of integral representations of Whittaker functions and their products (cf. H. BUCHHOLZ, *The confluent hypergeometric functions*, Springer Verlag, Berlin, 1969).

The most important tools in his investigations in this area are Barnes integrals for these functions which can be expressed in terms of generalized hypergeometric functions. In 1936 MEIJER [13] defined the G-function which is the most general useful function of this kind. If m, n, p, q are integers, $0 \leq n \leq p$, $0 \leq m \leq q$ then he defined

$$G^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right] \tag{1}$$

$$= \frac{1}{2\pi i} \int\limits_C \frac{\prod_{h=1}^{m} \Gamma(b_h - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{h=m+1}^{q} \Gamma(1 - b_h + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds,$$

where C is a contour in the complex s-plane from $\infty - i\alpha$ to $\infty + i\beta$ such that the poles of $\prod\limits_{h=1}^{m} \Gamma(b_h - s)$ are to the right of C and those of $\prod\limits_{j=1}^{n} \Gamma(1 - a_j + s)$ are to the left of C. In order that this last situation is possible the a_j and b_h and the real numbers α and β have to satisfy certain inequalities.

By means of residue calculus one shows that the G-function is a linear combination of generalized hypergeometric functions. In fact, this was Meijer's original definition of the G-function. The G-function satisfies the generalized hypergeometric differential equation and all significant particular solutions of this equation may be expressed in terms of the G-function.

BARNES (Proc. London Math. Soc. (2), 59-116 (1907)) considered the asymptotic behaviour of a fundamental system of solutions of the generalized hypergeometric differential equation, consisting of the G-functions corresponding to $m = 1$, $n = p$; $m = q$, $n = 1$; and $m = q$, $n = 0$. In his fundamental paper *On the G-function* of 1946 [34] MEIJER deduced the asymptotic expansions of G-functions in the case $p < q$ by expressing them in terms of the Barnes fundamental system of solutions. To this end he wrote the integrand in (1) as the product of the integrand for $m = q$, $n = 0$ and $\pi^{m-q+n} \left\{ \prod_{j=1}^{n} \sin \pi(a_j - s)\}^{-1} \prod_{h=m+1}^{q} \sin \pi(b_h - s) \right\}$ and expanded the last quotient as $e^{\lambda \pi i s} \sum_{j=1}^{n} c_j \{\sin \pi(a_j - s)\}^{-1} +$ a linear combination of terms $e^{\lambda_k \pi i s}$. Then one gets an expansion formula for the G-function in terms of the functions whose asymptotic behaviour has been given by BARNES. Similarly, he derived analytic continuations of the G-function in the case $p = q$.

The papers [5-28, 30-32, 35] contain integral relations for the functions of Bessel, Hankel, Legendre, Whittaker, Lommel and Struve and hypergeometric functions or products thereof which are derived from Barnes integrals. They are special cases of the following type integrals

$$\int_L e^{-t} t^{\alpha-1} G(zt) dt, \tag{2}$$

$$\int_L t^{\alpha-1} G(zt) G_1(\zeta t) dt, \tag{3}$$

$$\int_0^1 G(zt)\, {}_2F_1(\alpha, \beta; \gamma; 1-t)(1-t)^{\gamma-1} t^{\sigma} dt, \tag{4}$$

$$\int_1^{\infty} G(zt)\, {}_2F_1(\alpha, \beta; \gamma; 1-t)(t-1)^{\gamma-1} t^{\sigma} dt, \tag{5}$$

where G and G_1 are suitable G-functions and L is the positive real axis or a Hankel contour from ∞ to ∞ around $t = 0$, or a path from $t = -\infty i$ to $t = \infty i$. The evaluation of these integrals proceeds in general by substitution of the Barnes integral for $G(zt)$ and inverting the order of integration. The remaining integrals are either elementary as in (2) or can be calculated by expanding the hypergeometric function and using Gauss' formula for ${}_2F_1(1)$ as in the cases (4) and (5) or by using Mellin inversion theorem as in the second integral (3). The Mellin inversion theorem may be applied to the G-function in case the path in (1) may be replaced by a path from $-\infty i$ to $+\infty i$. In this last case one may also use the product formula for the Mellin transform.

In [33, 36, 37] MEIJER deduced several expansion formulae for the G-function. The simplest cases of these are Taylor expansions of $G(z)$ around a point $z = w$. For example one has

$$G^{m,n}_{p,q}\left[\lambda w \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{(\lambda-1)^r}{r!} G^{m,n+1}_{p+1,q+1}\left[w \,\middle|\, \begin{matrix} 0, a_1, \cdots, a_p \\ b_1, \cdots, b_q, r \end{matrix}\right], \quad (6)$$

if certain conditions are satisfied. MEIJER replaces λ by λt, multiplies both sides by $e^{-t}t^{\alpha-1}$ or $t^{\alpha-1}$ times a suitable generalized hypergeometric function ${}_k\phi_l(\pm t)$, and integrates both sides over a suitable contour in the t-plane using the results concerning (2)-(5). In this way he deduces expansions of G-functions $G(\lambda w)$ in terms of a sum of generalized hypergeometric functions ${}_h\phi_j(\lambda)$ times G-functions $G(w)$.

Many known relations for spacial functions, for example generating functions for orthogonal polynomials and generalized hypergeometric functions, are spacial cases of these expansion theorems.

A survey of Meijer's work on the G-function may be found in Vol. 1, Chapter 5, of *Higher transcendental functions*, Bateman Manuscript Project, editor A. ERDÉLYI, Mc Graw-Hill (1953). Extensions of Meijer's work have been given in recent years by several authors, in particular by Y.L. LUKE, J. FIELDS and J. WIMP on expansion theorems.

Meijer's name is also connected with two generalizations of the Laplace and Fourier transforms and their inversion theorems. He found these results in 1940 and 1941 [29], [32]. The first integral fransform of Meijer is connected with the Hankel transform. The corresponding inversion theorem gives conditions for the validity of the pair of formulae

$$f(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty K_\nu(st)(st)^{\frac{1}{2}} F(t)dt,$$

$$F(t) = \frac{1}{i\sqrt{2\pi}} \int_{\beta-i\infty}^{\beta+i\infty} I_\nu(ts)(ts)^{\frac{1}{2}} f(s)ds,$$

where K_ν is the Bessel function of the third kind and I_ν is a Bessel function with imaginary argument. The second integral transform of Meijer concerns the formulae

$$f(s) = \int_0^\infty e^{-\frac{1}{2}st} W_{k+\frac{1}{2},m}(st)(st)^{-k-\frac{1}{2}} F(t)dt,$$

$$F(t) = \frac{\Gamma(1-k+m)}{2\pi i \Gamma(1+2m)} \int_{\beta-\infty i}^{\beta+\infty i} e^{\frac{1}{2}ts} M_{k-\frac{1}{2},m}(ts)(ts)^{k-\frac{1}{2}} f(s)ds,$$

where W and M are Whittaker functions. Important special cases of these formulae are already given in previous papers by MEIJER. The proofs of these beautiful inversion theorems are related to Titchmarsh' treatment of the inversion theorem of Hankel. ERDÉLYI showed that these results may be deduced from the corresponding theorems for Laplace transforms by means of fractional integrations.

The last years of his life MEIJER worked on asymptotic expansions of G-functions with large parameters. Although he obtained many special results in this area, he delayed publication seeking a more complete and comprehensive treatment.

In his teaching as well as in his scientific work he was very scrupulous. His lectures were always well prepared and he devoted much of his time to students. He was very modest, much esteemed by his colleagues and students.

List of Papers by C.S. Meijer

[1] Asymtotische Entwicklungen von Besselschen, Hankelschen und verwandten Functionen. *Proc. Kon. Akad. Wet. Amsterdam* **35** (1932), 656-667, 852-866, 948-958, 1079-1090.

[2] Über die asymptotische Entwicklung von

$$\int_0^{\infty - i(\arg w - \mu)} e^{\nu z - w \sin hz} dz \quad (-\frac{\pi}{2} < \mu < \frac{\pi}{2})$$

für grosse Werte von $|w|$ und $|\nu|$. *Proc. Kon. Akad. Wet. Amsterdam* **35** (1932), 1170-1180, 1291-1303.

[3] Asymptotische Entwicklungen von Besselschen und Hankelschen Functionen für grosse Werte des Arguments und Ordung. *Math. Annalen* **108** (1933), 321-359.

[4] Asymptotische Entwicklungen Besselscher, Hankelscher und verwandter Functionen. Bestimmung von numerischen oberen Schranken für das Restgied mittels der Methode der Sattelpunkte. *Thesis*, Groningen, 1933.

[5] Über die Integraldarstellungen der Whittakerschen Funktion $W_{k,m}(z)$ und der Hankelschen und Besselschen Funktionen. *Nieuw Archief Wisk.* (2) **18** (1934), 35-37.

[6] Einige Integraldarstellungen für Whittakersche und Besselsche Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **37** (1934), 805-812.

[7] Noch einige Integraldarstellungen für die Whittakersche Funktion. *Proc. Kon. Akad. Wet. Amsterdam* **38** (1935), 528-535.

[8] Integraalvoorstellingen van producten van Hankelsche functies. *Handelingen XXVe Ned. Natuur- en Geneesk. Congres* , 1935.

[9] Integraldarstellungen aus der Theorie der Besselschen Funktionen. *Proc. Lond. Math. Soc.* (2) **40** (1936), 1-22.

[10] Einige Integraldarstellungen für Produkte von Whittakerschen Functionen. *Quart. J. Math. Oxf.* **6** (1935), 241-248.

[11] Integraldarstellungen für Lommelshe and Struvesche Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **38** (1935), 628-634, 744-749.

[12] Neue Integraldarstellungen aus der Theorie der Whittakerschen und Hankelschen Funktionen. *Math. Annalen* **112** (1936), 469-489.

[13] Über Wittakersche Bezw. Besselsche Funktionen und deren Produkte. *Nieuw. Archief Wisk.* (2) **18** (1936), 10-39.

[14] Einige Integraldarstellungen aus der Theorie der Besselschen und Whittakerschen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **39** (1936), 394-403, 519-527.

[15] Über Produkte von Whittakerschen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **40** (1937), 133-141, 259-262.

[16] Noch einige Integraldarstellungen für Produkte von Whittakerschen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **40** (1937), 871-879.

[17] Über eine Integraldarstellung der Whittaker Funktionschen. *Proc. Kon. Akad. Wet. Amsterdam* **41** (1938), 42-44.

[18] Einige Inegraldarstellungen für die Lommenshe Funktion $S_{\mu,\nu}(z)$. *Proc.*

Kon. Akad. Wet. Amsterdam **41** (1938), 151-154.

[19] Integraldarstellungen für Produkte von Legendreschen Funktionen. *Nieuw Archief Wisk.* (2) **19** (1938), 207-234.

[20] Note über das Produkt $M_{k,m}(z)M_{-k,m}(z)$. *Proc. Kon. Akad. Wet. Amsterdam* **41** (1938), 275-277.

[21] Beiträge zur Theorie der Whittakerschen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **41** (1938), 624-633, 744-755, 879-888.

[22] Über die Kummersche Funktion ${}_1F_1(a;b;z)$. *Proc. Kon. Akad. Wet. Amsterdam* **41** (1938), 1108-1114.

[23] Integraldarstellungen Whittakerscher Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **41** (1938), 1096-1107, **42** (1939), 141-146.

[24] Zur Theorie der hypergeometrischen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **42** (1939), 355-369.

[25] Über Besselsche, Lommelsche und Whittakershe Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **42** (1939), 872-879, 938-947.

[26] Über Produkte von Legendreschen Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **42** (1939), 930-937.

[27] Integraldarstellungen für Struvesche und Besselsche Funktionen. *Compositio Math.* **6** (1939), 348-367.

[28] Über Besselsche, Struvesche und Lommelsche Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **43** (1940), 198-210, 366-378.

[29] Über eine Erweiterung der Laplace-Transformation. *Proc. Kon. Akad. Wet. Amsterdam* **43** (1940), 599-608, 702-711.

[30] Neue Integralstellungen für Whittakersche Funktionen. *Proc. Kon. Akad. Wet. Amsterdam* **44** (1941), 81-92, 186-194, 298-307, 442-451, 590-598.

[31] Integraldarstellungen für Whittakersche Funktionen und ihre Produkte. *Proc. Kon. Akad. Wet. Amsterdam* **44** (1941), 435-441, 599-605.

[32] Eine neue Erweiterung der Laplace-Transformation. *Proc. Kon. Akad. Wet. Amsterdam* **44** (1941), 727-737, 831-839.

[33] Multiplikationstheoreme für die Funktion $G^{m,n}_{p,q}(z)$. *Proc. Kon. Akad. Wet. Amsterdam* **44** (1941), 1062-1070.

[34] On the G-function. *Proc. Kon. Akad. Wet. Amsterdam* **49** (1946), 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1165-1175.

[35] *Convergentie en divergentie.* Inaugular lecture. P. Noordhoff, Groningen, 1946.

[36] Neue Integraldarstellungen für Besselsche Funktionen. *Compositio*

Math. **8** (1951), 49-60.
[37] Expansion theorems for the G-function, *Proc. Kon. Akad. Wet. Amsterdam Ser. A* **55** (1952), 369-379, 483-487, **56** (1953), 43-49, 187-193, 349-357, **57** (1954), 77-82, 83-91, 273-279, **58** (1955), 243-251, 309-314, **59** (1956), 70-82.
[38] Ontwikkelingen van gegenraliseerde hypergeometrische functies. *Simon Stevin* **31** (1956), 117-139.

Theses Written Under Direction of C.S. Meijer

P. C. S i k k e m a, Differential operators and differential equations of infinite order with constant coefficients, Researches in connection with integral functions of finite order. P. Noordhoff, Groningen, 1953.
U. J. K n o t t n e r u s, Approximation formulae for generalized hypergeometric functions for large values of the parameters with applications to expansion theorems for the function $G^{m,n}_{p,q}(z)$. Wolters, Groningen, 1960.
B. L. J. B r a a k s m a, Asymptotic expansions and analytic continuations for a class of Barnes integrals. P. Noordhoff, Groningen, 1963.
W. K n o l, Generalizations of two relations in Bessel function theory. V. R. B. Groningen, 1970.

Mathematisch Instituut
Universiteit Groningen
Groningen, The NETHERLAND
e-mail: braaksma@math.rug.nl

Received (as Original Publication):
May 2, 1975
Permission for Reprinting:
March 26, 2002