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A NONLOCAL DIFFUSION EQUATION WITH SOURCE

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Abstract: We study the Cauchy problem for a nonlocal diffusion equation with source in the whole $\mathbb{R}^N$, $N \geq 1$. First, we prove existence and uniqueness and the validity of a comparison principle for the solutions. Next, we analyzed the phenomenon of blow-up in time finite for the solutions and study the blow-up rate for some sources given.

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1. Introduction

The mathematical description of a great variety of phenomena that appear in many sciences such as Biology, Chemistry, Physics, can be done using linear and nonlinear partial differential equations. Some of the models that appear are the so called diffusion models. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, smooth function, symmetric, radial function strictly decreasing, supported in unitary ball and $\int_{\mathbb{R}^N} J(r)dr = 1$. Equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t), \tag{1}$$

and variations of it, have been recently widely used to model diffusion processes,

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see [1], [3], [6]. The equation (1), so called nonlocal diffusion equation, since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x. This equation shares many properties with the classical heat equation $u_t = \Delta u$ such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed.

A simple nonlocal nonlinear model in one dimensional for diffusion where the diffusion at a point depends on the density, as happens for the porous medium equation, was introduced in [4]. In [2] extend this model in higher space dimensions. In this model the probability distribution of jumping from location y to location x is given for all $0 < \alpha \leq \frac{1}{N}$ by $J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)\frac{1}{u^{N\alpha}(y,t)}$ when $u(y,t) > 0$ and 0 otherwise. In this case the rate at which individuals are arriving to position x from all other places is $\int_{\mathbb{R}^N} J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)u^{1-N\alpha}(y,t)dy$ and the rate at which they are leaving location x to travel to all other sites is $-u(x,t) = -\int_{\mathbb{R}^N} J\left(\frac{y-x}{u^{\alpha}(x,t)}\right)u^{1-N\alpha}(x,t)dy$. As before this consideration, in the absence of external sources, leads immediately to the fact that the density u has to satisfy

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)u^{1-N\alpha}(y,t)dy - u(x,t), \tag{2}$$

with initial data $u(\cdot,0) = d + w_0(x)$, $w_0 \in L^1(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$ non-negative and $d \geq 0$.

It is proved in [2] that this problem shares with the porous medium equation the finite speed of propagation property. Compactly supported initial data develop a free boundary. Moreover, we study this equation in a domain smooth $\Omega \in \mathbb{R}^N$ with homogeneous Neumann boundary conditions and prove that solutions exist globally and stabilize to the mean value of the initial data as $t \to \infty$. We study also the case with homogeneous Dirichlet boundary conditions and prove that solutions exist globally and stabilize to zero as $t \to \infty$.

One of the most remarkable properties that distinguish nonlinear evolution problems concern by the study of solutions non bounded, such phenomenon is known like $Blow-up$. A solution blows-up in finite time if there exists a time $0 < T < \infty$, such that the solution is well defined for all $t \in [0,T)$, but $\sup_{x\in\Omega} u(x,t) \to \infty \ as \ t \to T^-$. For more details, see [5], [7], [8].

The purpose of this paper is to present a variation of the model given by the problem (2), we consider the presence of external sources, we obtain the following Cauchy problem for $\mathbb{R}^N \times [0,\infty)$,

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{u^\alpha(y,t)}\right) u^{1-N\alpha}(y,t)dy - u(x,t) + f(u(x,t)) \tag{3}$$

with initial condition $u(x,0) = u_0(x)$, where $u_0 \in L^1(\mathbb{R}^N)$ non-negative function and f is a function of u representing reaction (source). We considered the following hypothesis in f:

(H_1): $f : [0,\infty) \longrightarrow [0,\infty)$, increasing function, $f(0) \geq 0$, $f(s) > 0$ for all $s > 0$.

(H_2): $\displaystyle\int^\infty \frac{1}{f(s)} ds < \infty$.

The paper is organized as follows: in Section 2 we prove existence and uniqueness and a comparison principle of (3). In Section 3 we study the blow-up phenomena for solutions of problem (3).

2. The Cauchy Problem. Existence and Uniqueness

First, we study the problem for $(x,t) \in \mathbb{R}^N \times [0,\infty)$

$$u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{u^\alpha(y,t)}\right) u^{1-N\alpha}(y,t)dy - u(x,t) + f(u(x,t)) \tag{4}$$

with $u(x,0) = u_0(x)$, where $u_0 \in L^1(\mathbb{R}^N)$ non-negative function and f is a Lipschitz function. Then a convergence argument will extend our study to a function that satisfies H_1.

The existence and uniqueness of the solution for problem (4) result will be a consequence of Banach's fixed point theorem. Fix $t_0 > 0$ and consider the Banach space $X = C([0,t_0]; L^1(\mathbb{R}^N))$ with the norm $|||w||| = \max_{0\leq t\leq t_0} \|w(\cdot,t)\|_{L^1(\mathbb{R}^N)}$. Let $X_{t_0} = \left\{w \in C([0,t_0]; L^1(\mathbb{R}^N)) \;/\; w \geq 0\right\}$ which is a closed subset of $C([0,t_0]; L^1(\mathbb{R}^N))$. We will obtain the solution as a fixed point of the operator $T_{w_0} : X_{t_0} \to X_{t_0}$ defined by

$$\begin{aligned} T_{w_0}(w)(x,t) = & \int_0^t e^{s-t} \int_{\mathbb{R}^N} J\left(\frac{x-y}{w(y,s)^\alpha}\right) w(y,s)^{1-N\alpha} dyds \\ & + \int_0^t e^{s-t} f(w(x,s))ds + e^{-t} w_0(x). \end{aligned}$$

The following lemma is very important for our study.

Lemma 2.1. *Let f be Lipschitz function with Lipschitz's constant $K > 0$, w_0, z_0 non negative functions such that $w_0, z_0 \in L^1(\mathbb{R}^N)$ and $w, z \in X_{t_0}$, then there exists a constant $C = C(t_0, K) > 0$ such that*

$$|||T_{w_0} - T_{z_0}||| \leq C|||w - z||| + ||w_0 - z_0||_{L^1(\mathbb{R}^N)}.$$

Proof. Let $w, z \in X_{t_0}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |T_{w_0}(w)(x,t) - T_{z_0}(z)(x,t)|\,dx \\
&\leq \int_0^t e^{s-t} \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \left| J\left(\frac{x-y}{w(y,s)^\alpha}\right) w(y,s)^{1-N\alpha} \right.\right. \\
&\qquad\qquad\qquad \left.\left. - J\left(\frac{x-y}{z(y,s)^\alpha}\right) z(y,s)^{1-N\alpha} \right| dy \right] dx\,ds \\
&+ \int_0^t e^{s-t} \int_{\mathbb{R}^N} |f(w(x,s)) - f(z(x,s))|dx\,ds + e^{-t} \int_{\mathbb{R}^N} |w_0 - z_0|(y)dy \\
&= I_1 + I_2 + e^{-t} \int_{\mathbb{R}^N} |w_0 - z_0|(y)dy.
\end{aligned}$$

To analyze I_1, now set $A^+(s) = \left\{y \in \mathbb{R}^N / w(y,s) \geq z(y,s)\right\}$ and $A^-(s) = \left\{y \in \mathbb{R}^N / w(y,s) < z(y,s)\right\}$. Since J is radial function strictly decreasing, we have that the integrands are non negative and we can apply Fubini's theorem. Therefore,

$$I_1 = \int_0^t e^{s-t} \int_{\mathbb{R}^N} |w(y,s) - z(y,s)|dy\,ds.$$

For I_2, as f is Lipschitz, we to get

$$I_2 \leq K \int_0^t e^{s-t} \int_{\mathbb{R}^N} |w(y,s) - z(y,s)|dy\,ds.$$

In summary,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |T_{w_0}(w)(x,t) - T_{z_0}(z)(x,t)|\,dx \\
&\leq (1+K) \int_0^t e^{s-t} \int_{\mathbb{R}^N} |w(y,s) - z(y,s)|dyds + e^{-t} \int_{\mathbb{R}^N} |w_0 - z_0|(y)dy.
\end{aligned}$$

Hence, we get $|||T_{w_0} - T_{z_0}||| \leq C|||w - z||| + ||w_0 - z_0||_{L^1(\mathbb{R}^N)}$, where $C = (1+K)(1 - e^{-t_0})$, as desired. □

Next, we prove a theorem of existence and uniqueness.

Theorem 2.1. *For all $u_0 \in L^1(\mathbb{R}^N)$ non-negative function and for f Lipschitz function with Lipschitz's constant K there exists a unique solution u of (4) such that $u \in X_{t_0}$.*

Proof. Now taking $z_0 \equiv 0$ and $z \equiv 0$ in Lemma 2.1 we get that $T_{w_0} \in X_{t_0}$. Moreover, taking $z_0 \equiv w_0$ in Lemma 2.1 and $C = (1+K)(1-e^{-t_0}) < 1$, we get that T_{w_0} is a strict contraction in X_{t_0}, therefore there exists a unique fixed point of T_{u_0} in X_{t_0} for Banach's fixed point theorem. This provides us with a solution of (4) in $[0, t_0]$. To continue we may take $u(x, t_0)$ as initial data and obtain a solution in $[0, 2t_0]$. We continue this procedure to obtain a solution of (4) defined for all $t > 0$. □

Now we give some consequences stated as remarks.

Remark 2.1. Solutions of (4) depend continuously on the initial data. In fact if u and v are solutions of (4) with initial data u_0 and v_0 respectively, then there exists a constant $\widetilde{C} = \widetilde{C}(t_0, K)$ such that

$$|||u(\cdot,t) - v(\cdot,t)||| \le \widetilde{C}\|u_0 - v_0\|_{L^1(\mathbb{R}^N)}.$$

Remark 2.2. The function u is solution of (4) if and only if

$$u(x,t) = \int_0^t e^{s-t} \int_{\mathbb{R}^N} J\left(\frac{x-y}{u^\alpha(y,s)}\right) u^{1-N\alpha}(y,s)dy\,ds + \int_0^t e^{s-t} f(u(x,s))ds + e^{-t}u_0(x).$$

Theorem 2.2. (Comparison Principle) *Let u and v be two continuous solutions of (4) with initial data u_0 and v_0 respectively. If $u(x,0) \le v(x,0)$ for $x \in \mathbb{R}^N$, then $u(x,t) \le v(x,t)$ for all $(x,t) \in \mathbb{R}^N \times [0,T)$.*

Proof. We assume that $u(x,0)$ and $v(x,0)$ are compactly supported C^1 functions, therefore there exist $\delta > 0$ such that $u(x,0) + \delta < v(x,0)$. Assume, for a contradiction that the conclusion does not hold. There exists a time $t_0 > 0$ and a point $x_0 \in \mathbb{R}^N$ such that $u(x_0,t_0) = v(x_0,t_0)$ and $u(x,t) \le v(x,t)$ for all $(x,t) \in \mathbb{R}^N \times [0,t_0]$. Let us consider the set $G = \{x \in \mathbb{R}^N / \ u(x,t_0) = v(x,t_0)\}$.

$G \neq \emptyset$ since $x_0 \in G$, moreover G is closed. Sea $x_1 \in G$, we have then

$$0 \leq (u-v)_t(x_1,t_0) =$$

$$\int_{\mathbb{R}^N} \left(J\left(\frac{x_1-y}{u^\alpha(y,t_0)}\right) u^{1-N\alpha}(y,t_0) - J\left(\frac{x_1-y}{v^\alpha(y,t_0)}\right) v^{1-N\alpha}(y,t_0)\right) dy \leq 0,$$

which implies $u(y,t_0) = v(y,t_0)$ for all $y \in B_\epsilon(x_1)$, hence G is open. If follows that $G = \mathbb{R}^N$ which is the desired contradiction.

We now get rid of the extra hypothesis that $w(x,0)$ and $z(x,0)$ are compactly supported C^1 functions. In order to do this let $u_n(x,0)$ and $v_n(x,0)$ be sequences of compactly supported C^1 functions such that $u_n(x,0) \to u(x,0)$ and $v_n(x,0) \to v(x,0)$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$ and, moreover, $u_n(x,0) \leq v_n(x,0)$. Let u_n and v_n be that solutions of (3) with initial data $u_n(x,0)$ and $v_n(x,0)$ respectively. By the previous argument one has $u_n(x,t) \leq v_n(x,t)$. The result follows by letting $n \to \infty$ in view of Remark 2.2 and the monotone convergence theorem. □

Remark 2.3. The Comparison principle is valid in L^1.

Theorem 2.3. *For all $u_0 \in L^1(\mathbb{R}^N)$ non-negative function, for all f that it satisfies (H_1) there exist a time $T > 0$ and a unique solution u of (3) such that $u \in X_{t_0}$.*

Proof. Let $(f_n)_n$ a sequence of Lipschitz creasing functions such that $f_n \leq f_{n+1}$. Moreover, assume that $f_n(s) = f(s)$ in $[0,n]$. Let u_n a unique solution of (4) with source f_n and initial data $u_n(x,0)$. Assume that initials data satisfy $u_n(x,0) < u_{n+1}(x,0)$ and $u_n(\cdot,0)$ uniformly convergent to $u(\cdot,0)$. By Comparison Principle Theorem 2.2, it is had that $u_n(x,t) \leq u_{n+1}(x,t)$, hence there exist u that it can be ∞ in some points such that $\lim_{n\to\infty} u_n = u$. Let $T = \sup\{t \mid \sup_{x\in\mathbb{R}^N} u(x,t) < \infty\}$, with $T > 0$. Like before if in the integral equation of Remark 2.2 we do $n \to \infty$, then after an application of the monotone convergence theorem, it follows that u is a unique solution of (3) in $\mathbb{R}^N \times [0,T)$ with initial data $u(x,0) = u_0(x)$ and source $f(u)$. □

In a similar way, we have the Comparison Principle Theorem for functions f satisfying H_1.

Theorem 2.4. (Comparison Principle) *Let f be a function that satisfies assumption H_1, u and v be continuous solutions of (3) with initial data u_0 and*

v_0 *respectively . If* $u(x,0) \le v(x,0)$ *for all* $x \in \mathbb{R}^N$*,, then* $u(x,t) \le v(x,t)$ *for all* $(x,t) \in \mathbb{R}^N \times [0,T)$*.*

3. Blow-Up Analysis

Next, we study the behavior of the solutions of (3).

Theorem 3.1. *Let* u *be a solution of (3) with initial data* $u_0 > 0$ *and source* f *satisfying assumption* (H_2)*, then* u *blows-up in finite time.*

Proof. Since the initial datum is positive, there exists $\delta > 0$ such that $u_0 \ge \delta > 0$. Let $v(t)$ be a solution of

$$v'(t) = f(v(t)) \;\; t > 0 \;\text{ and }\; v(0) = \delta > 0. \tag{5}$$

Since $v(t) = \displaystyle\int_{\mathbb{R}^N} J\left(\frac{x-y}{v^\alpha(t)}\right) v^{1-N\alpha}(t)dy$, we to get $v(t)$ is solution of (3) with initial data $v(0) = \delta > 0$.

We have of (5) that $\displaystyle\int_\delta^{v(t)} \frac{1}{f(s)}ds = t$. From (H_2), we have that exist a $T > 0$ such that $v(t)$ is defined on a finite time interval $[0,T)$ and $v(t) \to \infty$, as $t \to T^-$. Now, let u a solution of (3) with data initial u_0, since $u_0 \ge \delta = v(0) > 0$ then by Comparison Principle we have that $u(x,t) \ge v(t)$ for all $x \in \mathbb{R}^N$ and for all $t > 0$, therefore u blows-up in finite time $T > 0$, moreover $u(x,t)$ blows-up on all of the space $\mathbb{R}^N$ at the same time. □

Corollary 3.1. *Let* u *be solution of (3) with* $f(u) = u^p,\ p > 1;\ f(u) = e^u;\ f(u) = (1+u)\ln^p(1+u),\ p > 1$*, then* u *blows-up in finite time.*

Definition 3.1. A function $\overline{u} \in X_{t_0}$ is a super-solution of (3) if $\overline{u}(x,0) \ge u_0(x)$ and

$$\overline{u}_t(x,t) \ge \int_{\mathbb{R}^N} J\left(\frac{x-y}{\overline{u}^\alpha(y,t)}\right) \overline{u}^{1-N\alpha}(y,t)dy - \overline{u}(x,t) + f(\overline{u}(x,t)).$$

Theorem 3.2. *(Blow-up rates) Let u be solution of (3) with $f(u) = u^p,\ p > 1$ and blow-up in finite time T, then*

$$\lim_{t \to T^-} (T-t)^{1/(p-1)} \max_{x \in \mathbb{R}^N} u(x,t) = C_p,\ \textit{with}\ C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}.$$

Proof. We first show that $\max_{x \in \mathbb{R}^N} u(x,t) \geq C_p(T-t)^{-1/(p-1)}$. Assume, for a contradiction that the conclusion does not hold, there exists a time $t_0 > 0$ such that $\max_{x \in \mathbb{R}^N} u(x,t_0) < C_p(T-t_0)^{-1/(p-1)}$. We can choose $\widetilde{T}$ near to T with $\widetilde{T} > T$ such that $\max_{x \in \mathbb{R}^N} u(x,t_0) < C_p(\widetilde{T}-t_0)^{-1/(p-1)}$. Let $z(t) = C_p(\widetilde{T}-t)^{-1/(p-1)}$ a solution of (3), moreover $z(t_0) \leq \max_{x \in \mathbb{R}^N} u(x,t_0)$, then by Comparison Principle we have that $z(t) \leq \max_{x \in \mathbb{R}^N} u(x,t)$ for all $\mathbb{R}^N \times [t_0, T)$, then u blows-up in finite time $T^* > \widetilde{T}$, which is the desired contradiction.

From now prove the opposite inequality. Consider $\overline{u}(x,t) = C_p(T-t)^{-1/(p-1)}$, with $\overline{u}(x,0) \geq u_0(x)$ a super-solution of (3). Then by Comparison Principle we have that $\overline{u}(x,t) \geq u(x,t)$ for all $x \in \mathbb{R}^N$ and for all $t > 0$, therefore $\max_{x \in \mathbb{R}^N} u(x,t) \leq C_p(T-t)^{-1/(p-1)}$. □

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