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ON THE CHROMATIC POLYNOMIAL OF A CYCLE GRAPH

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Abstract: The aim of this article is to study the chromatic polynomial of cycle graph, and to describe some algebraic properties about the chromatic polynomial's coefficients and roots to the same graph.

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1. Introduction

During the period that the Four Color Problem was still unsolved, which spanned more than a century, many approaches were introduced with the hope that they would lead to a solution of this famous problem. In 1912 George David Birkhoff defined a function $P(M, \lambda)$ that gives the number of proper λ-colorings of a map M for a positive integer λ. As we will see, $P(M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M. Consequently, if it could be verified that $P(M, 4) > 0$ for every map M, then this would have established the truth of the Four Color Conjecture.

In 1932 Hassler Whitney expanded the study of chromatic polynomials from maps to graphs. While Whitney obtained a number of results on chromatic polynomials of graphs and others obtained results on the roots of chromatic polynomials of planar graphs, this did not contribute to a proof of the Four Color Conjecture.

Renewed interest in chromatic polynomials of graphs occurred in 1968 when Ronald C. Read wrote a survey paper on chromatic polynomials [6].

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There are several results about the zero distribution of chromatic polynomials on the real line and in the complex plane, see [1],[2],[3] and [9].

Finally, we will focus on the paper given by Meredith [7], and re-prove some of theories contained in Tang [5].

2. Preliminaries

All graphs considered in this paper are assumed to be finite and loopless. Let G be a graph with a positive integer λ, the number of different proper λ-colorings of G is denoted by $P(G, \lambda)$ and is called the chromatic polynomial of G.

A graph G is k-colorable if there exists a coloring of G from a set of k colors. In other words, G is k-colorable if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the chromatic number of G and is denoted by $\chi(G)$.

Two graphs are chromatically equivalent if they have the same chromatic polynomial. So two chromatically equivalent graphs must have the same order, the same size, and the same chromatic number. A graph G is chromatically unique if $P(K, \lambda) = P(G, \lambda)$ implies that $K \cong G$.

Theorem 1. (see [6]) *The chromatic polynomial $P(G, \lambda)$ of a graph G is a polynomial in λ.*

Proof. See [6]. □

Corollary 1. (see [6]) *$P(G, \lambda)$ has a degree of $n = |V(G)|$.*

Corollary 2. (see [6]) $P(G, 0) = 0$.

3. The Chromatic Polynomial of a Cycle Graph

A cycle graph is a graph which consists of a single cycle. We denote the cycle graph by C_n. In addition, the number of vertices in C_n equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.

Result 1. (see [10]) *We prove that for the cycle C_n of order n, the chromatic polynomial is:*

(i) $P(C_2, \lambda) = \lambda(\lambda - 1)$, *for* $n \geq 2$.

(ii) $P(C_2, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$, *for* $n > 2$.

Proof. (Induction on n) For $n = 2$, its clear.
For $n = 3$, observe that $C_3 = K_3$, so

$$\begin{aligned} P(C_3, \lambda) &= (\lambda - 1)^3 + (-1)^3(\lambda - 1) \\ &= \lambda^{-}3\lambda^2 + 2\lambda \\ &= \lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

For $n \geq 4$, observe that $P(C_n, \lambda) = P(P_{n-1}, \lambda) - P(C_{n-1}, \lambda)$. We know that $P(P_n, \lambda) = \lambda(\lambda - 1)^n$, where P_n is path graph of n vertices. Thus

$$\begin{aligned} P(C_n, \lambda) &= P(P_{n-1}, \lambda) - P(C_{n-1}, \lambda) \\ &= \lambda(\lambda - 1)^{n-1} - [(\lambda - 1)^{n-1} + (-1)^{n-1}(\lambda - 1)] \\ &= (\lambda - 1)^n + (-1)^n(\lambda - 1). \end{aligned}$$

□

Example 2.1. (see [6]) Determine the chromatic polynomial of C_4 in Figure 3.1. There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from that assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda - 1)^2$.

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4, the number of λ-colorings of C_4 in which v_2 and v_4 are colored differently is $\lambda(\lambda - 1)(\lambda - 2)^2$. Hence the number of distinct λ-colorings of C_4 is

$$\begin{aligned} P(C_4, \lambda) &= \lambda(\lambda - 1)^2 + \lambda(\lambda - 1)(\lambda - 2)^2 \\ &= \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda - 1)^4 + (\lambda - 1). \end{aligned}$$

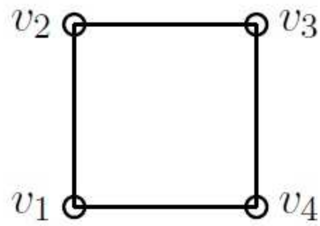

Figure 3.1: The chromatic polynomial of C_4

Identity 1. (see [5]) *For $n > 2$ and $\lambda \in \mathbb{R}$,*

$$\begin{aligned}\sum_{i=0}^{n-1}(-1)^i\binom{n}{n-i}\lambda^{n-i}+(-1)^n\lambda &= \sum_{i=0}^{n-2}(-1)^i\lambda(\lambda-1)^{n-i-1}\\ &= (\lambda-1)^n+(-1)^n(\lambda-1)\\ &= P(C_n,\lambda),\end{aligned}$$

where $P(C_n,\lambda)$ is the chromatic polynomial of a cycle with n vertices.

Proof. See [5]. □

Proposition 1. (see [6]) *For every integer $n \geq 3$,*

$$\chi(C_n)=\begin{cases}2 & , \quad n \text{ is even,}\\ 3 & , \quad n \text{ is odd.}\end{cases}$$

Corollary 3. (see [8]) *Let G be a graph. Then $\chi(G) \geq 3$ if and only if G has an odd cycle.*

Remark 1. (see [4]) All Cycle graphs are chromatically unique.

In this part we will summarize some algebraic properties about the chromatic polynomial's coefficients and roots as well as their relations to cycle graph-theoretic properties of G.

This part focuses on a paper written by Meredith [7]. The paper gave a couple of theorems that give the coefficients of chromatic polynomials, one for a graph of n vertices and k edges (Theorem 1 in [7]), and the latter, for a graph with one cycle of a specific length (Theorem 3 in [7]).

Theorem 2. (see [5]) *If a connected graph G has n vertices, k edges and if the coefficient of λ^r of $P(G,\lambda)$, the chromatic polynomial of G, is α_r, then* $|\alpha_r| \leq \binom{\kappa}{n-r}$.

Proof. Given a graph G with n vertices and k edges, to find α_r, we would need to contract and delete to an r null graph. To do so, we must contract exactly $n-r$ edges, since a contraction is the only way to reduce vertices, which can be done in $\binom{\kappa}{n-r}$ ways. However, some graphs contract more than one edge when doing a single contraction step. For example, a 3-cycle contracts 2 edges in one step. Hence some graphs have fewer than $\binom{\kappa}{n-r}$ contractions that can be realized. Therefore, $\alpha_r \leq \binom{\kappa}{n-r}$. □

Theorem 3. (see [5]) *Let G be a connected graph with n vertices and one cycle of order x. Then $P(G_n,\lambda) = \sum_{i=1}^{x-1}(-1)^{i-1}\lambda(\lambda-1)^{n-i}$. Furthermore, if α_r is the coefficient for λ^r, then*

$$|\alpha_r| = \binom{n}{r} - \binom{n-x+1}{r}.$$

Proof. Let G be a graph with n edges and one cycle of length x. First, we want to calculate $P(G_n,\lambda)$, by deleting and contracting to trees from this graph. To do this, we can use the deletion-contraction algorithm on the x-cycle. A deletion of one of the edges on the x-cycle, would leave us with an n-tree, which has a chromatic polynomial of $\lambda(\lambda-1)^{n-1}$. Looking at the contraction step, we are left with a graph with $n-1$ vertices and exactly one $x-1$-cycle. If we do a deletion, we would get an $n-1$-tree, with the chromatic polynomial of $\lambda(\lambda-1)^{n-2}$. A contraction would leave us with a graph with $n-2$ vertices and exactly one $x-2$-cycle. If we do k contractions and a deletion, $0 \leq k \leq n-3$, we get a $(n-k)$-tree with the chromatic polynomial of $\lambda(\lambda-1)^{n-k-1}$. Looking at the 3 cycle now, a deletion from the 3-cycle would leave us with an $(n-x+3)$-tree, $\lambda(\lambda-1)^{n-x+2}$, and a contraction would give us an $(n-x+2)$-tree, $\lambda(\lambda-1)^{n-x+1}$.

Combining the pieces of chromatic polynomials we found from the trees in the graph and remembering that contractions give negative addends, we get,

$$\begin{aligned} P(G_n,\lambda) &= \lambda(\lambda-1)^{n-1} - \lambda(\lambda-1)^{n-2} + \cdots + (-1)^{x-3}\lambda(\lambda-1)^{n-x+2} \\ &+ (-1)^{x-2}\lambda(\lambda-1)^{n-x+1} \\ &= \sum_{i=1}^{x-1}(-1)^{i-1}\lambda(\lambda-1)^{n-i}. \end{aligned}$$

To get α_r, we will look at our previous result. By the binomial theorem, we get that,

$$\sum_{i=1}^{x-1}(-1)^{i-1}\lambda(\lambda-1)^{n-i} = \sum_{i=1}^{x-1}\sum_{j=0}^{n-i}(-1)^{n-j}\binom{n-i}{j}\lambda^{j+1}. \tag{1}$$

Using the Hockey Stick Lemma, which says

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{n-2}{m}+\binom{n-1}{m}=\binom{n}{m+1}, m\in\mathbb{N},$$

we can sum up the coefficients of each λ^r. However, the coefficients stop at a certain point, in our case, when we have $\binom{n-x+1}{m}$. Taking the coefficient of λ^i in Equation (1), we have, by the Hockey Stick lemma,

$$\begin{aligned} |\alpha_r| &= \binom{n-1}{r-1}+\binom{n-2}{r-1}+\binom{n-3}{r-1}+\cdots+\binom{n-x+2}{r-1} \\ &\quad+\binom{n-x+1}{r-1} \\ &= \binom{n-1}{r-1}+\binom{n-2}{r-1}+\cdots+\binom{2}{r-1}+\binom{1}{r-1} \\ &\quad-\left[\binom{n-x}{r-1}+\binom{n-x+1}{r-1}+\cdots+\binom{2}{r-1}+\binom{1}{r-1}\right]. \end{aligned}$$

So $|\alpha_r| = \binom{n}{r} - \binom{n-x+1}{r}$.

In Meredith's paper [7], his third theorem states that if G has just one circuit, of length $n-s+1$, then:

(a) $|\alpha_r| = \binom{k}{n-r}$, if $r > s$.

(b) $|\alpha_r| = \binom{k}{n-r} - \binom{k-n+s}{s-r}$, if $r \leq s$,

where k is the number of edges and n is the number of vertices.

Note that in a graph with only one cycle, the total number of vertices, n, and edges, k, are the same (i.e. add an edge to a tree). So we can actually replace k by n. Looking at case (a), we can change that to $|\alpha_r| = \binom{n}{n-r}$, and case (b) to $\alpha_r = \binom{n}{n-r} - \binom{s}{s-r}$.

Also note that in a combination, $\binom{p}{q}$, if $p < q$, then $\binom{p}{q} = 0$. With that, we can combine cases (a) and (b) to $|\alpha_r| = \binom{n}{n-r} - \binom{s}{s-r} = \binom{n}{r} - \binom{n-x+1}{r}$, since $n-s+1 = x$, we get that $s = n-x+1$. So now we can see Meredith's third theorem is the same as Theorem 3.2 in terms of $|\alpha_r|$, but we also give a formula for $P(G, \lambda)$ in terms of $\lambda(\lambda-1)$.

Meredith's proof for his third theorem counts the number of spanning subgraphs in his original graph to get the $|\alpha_r|$. In doing so, he used five different variables which ended up very confusing to follow. Our proof is just a straight forward calculation using the contraction and deletion algorithm, and counting the number of ways we can do each. $\square$

Definition 1. A chromatic root of a graph G is a zero of the chromatic polynomial of G.

It follows that $k \in N$ is a root if and only if $k < \chi(G)$.

Theorem 4. (see [2]) *The chromatic polynomial has no real roots in the interval* $\left(1, \frac{32}{27}\right]$.

Consequence 3.1. (see [2], [9]) There are no real roots in $(-\infty, 0)$, $(0, 1)$.

Theorem 5. (see [3]) *The real roots of all chromatic polynomials are dense in* $\left[\frac{32}{27}, \infty\right)$.

Theorem 6. (see [1]) *The roots of all chromatic polynomials are dense in* $\mathbb{C}$.

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