RESULTS ON THE BETA FUNCTION
AND THE INCOMPLETE BETA FUNCTION

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Abstract: The incomplete Beta function $B(\lambda, \mu; x)$ is defined by

$$B(\lambda, \mu; x) = \int_0^x t^{\lambda-1} (1-t)^{\mu-1} dt,$$

for $\lambda, \mu > 0$ and $0 < x < 1$. This definition was extended to negative integer values of λ and μ by Özçağ et al. Partial derivatives of the incomplete Beta function $B(\lambda, \mu; x)$ for negative integer values of λ and μ were then evaluated. In the following, it is proved that

$$B_{p,0}(\lambda, 1; x) = - \sum_{i=1}^{p+1} \frac{(-1)^i p!}{(p-i+1)! \lambda^i} x^\lambda \ln^{p-i+1} x$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$ and more generally

$$B_{p,0}(\lambda, s; x) = \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j B_{p,0}(\lambda + j, 1; x)$$

for $p = 0, 1, 2, \dots$ and $s = 1, 2, \dots$. Further results are also proved.

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The Beta function $B(\lambda, \mu)$, see Sneddon [6], is usually defined by the integral

$$B(\lambda, \mu) = \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} dt, \quad (1)$$

for $\lambda, \mu > 0$ and more generally, the function

$$\frac{\partial^{p+q}}{\partial \lambda^p \partial \mu^q} B(\lambda, \mu) = B_{p,q}(\lambda, \mu)$$

is defined by the integral

$$B_{p,q}(\lambda, \mu) = \int_0^1 t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q (1-t) dt,$$

for $\lambda, \mu > 0$ and $p, q = 0, 1, 2, \dots$.

It follows from equation (1) that

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda + \mu)},$$

where $\Gamma(\lambda)$ denotes the Gamma function and this equation is then used to define $B(\lambda, \mu)$ for $\lambda, \mu < 0$ and $\lambda, \mu \neq -1, -2, \dots$.

More generally, it was proved in [2] that the neutrix limit

$$B_{p,q}(\lambda, \mu) = N\text{-}\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1-\epsilon} t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q (1-t) dt, \quad (2)$$

exists for all λ, μ and $p, q = 0, 1, 2, \dots$, where N is the neutrix, see van der Corput [1], having domain $N' = (0, x)$ ($0 < x < 1/2$) and range N'' the real numbers, with the negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \quad \ln^r \epsilon \quad (\lambda < 0, \quad r = 1, 2, \dots)$$

and all functions which converge to zero in the normal sense as ϵ tends to zero. Equation (2) was then used to define $B_{p,q}(\lambda, \mu)$ for all λ, μ and $p, q = 0, 1, 2, \dots$.

Note that if $\lambda > 0$, we could write

$$B_{p,q}(\lambda, \mu) = N\text{-}\lim_{\epsilon \rightarrow 0} \int_0^{1-\epsilon} t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q (1-t) dt,$$

and if $\mu > 0$, we could write

$$B_{p,q}(\lambda, \mu) = \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^1 t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q(1-t) dt.$$

The incomplete Beta function $B(\lambda, \mu; x)$ is defined by

$$B(\lambda, \mu; x) = \int_0^x t^{\lambda-1} (1-t)^{\mu-1} dt, 0 < x < 1$$

for $\lambda, \mu > 0$ and $0 < x < 1$, see Özçağ et al. [4].

It was proved, see Özçağ et al. [4] and [5] that

$$\text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} (1-t)^{\mu-1} dt$$

exists for all values of λ, μ and so $B(\lambda, \mu; x)$ was defined by the equation

$$B(\lambda, \mu; x) = \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} (1-t)^{\mu-1} dt$$

for all values of λ, μ .

More generally, the function $B_{p,q}(\lambda, \mu; x)$ is defined by

$$B_{p,q}(\lambda, \mu; x) = \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q dt,$$

the limit existing for all values of λ, μ and $p, q = 0, 1, 2, \dots$, see [5].

Note that if $\lambda > 0$, then we simply have

$$\begin{aligned} B_{p,q}(\lambda, \mu; x) &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q(1-t) dt \\ &= \int_0^x t^{\lambda-1} \ln^p t (1-t)^{\mu-1} \ln^q(1-t) dt. \end{aligned}$$

Note further that

$$B_{p,q}(\lambda, \mu) = \text{N-lim}_{\epsilon \rightarrow 0} B_{p,q}(\lambda, \mu; 1 - \epsilon), \quad (3)$$

for all values of λ and μ and in particular

$$B_{p,q}(\lambda, \mu) = B_{p,q}(\lambda, \mu; 1), \quad (4)$$

if $\mu > 0$.

To prove our first theorem, we need the following result:

$$\begin{aligned} & \int_{\epsilon}^x t^{\lambda-1} \ln^p t \, dt \\ &= - \sum_{i=1}^{p+1} \frac{(-1)^i p!}{(p-i+1)! \lambda^i} x^{\lambda} \ln^{p-i+1} x + \sum_{i=1}^{p+1} \frac{(-1)^i p!}{(p-i+1)! \lambda^i} \epsilon^{\lambda} \ln^{p-i+1} \epsilon, \end{aligned} \quad (5)$$

for $\lambda \neq -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

Equation (5) is easily proved by induction.

Theorem 1.

$$B_{p,0}(\lambda, 1; x) = - \sum_{i=1}^{p+1} \frac{(-1)^i p!}{(p-i+1)! \lambda^i} x^{\lambda} \ln^{p-i+1} x \quad (6)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

More generally,

$$B_{p,0}(\lambda, s; x) = \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j B_{p,0}(\lambda + j, 1; x) \quad (7)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Proof. We have

$$B_{p,0}(\lambda, 1; x) = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} \ln^p t \, dt$$

and equation (6) follows on using equation (5) for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

To prove equation (7), we have

$$\int_{\epsilon}^x t^{\lambda-1} \ln^p t (1-t)^{s-1} \, dt = \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \int_{\epsilon}^x t^{\lambda+j-1} \ln^p t \, dt$$

and equation (7) follows on using equations (5) and (6) for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Corollary 1.1

$$B_{p,0}(\lambda, 1) = \frac{(-1)^p p!}{\lambda^{p+1}} \quad (8)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$

More generally

$$B_{p,0}(\lambda, s) = \sum_{j=0}^{s-1} \binom{s-1}{j} \frac{(-1)^{p+j} p!}{(\lambda+j)^{p+1}} \quad (9)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$

Proof. Equation (8) follows immediately from equations (4) and (6) for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$

Equation (9) follows from equations (4), (7) and (8) for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$

Theorem 2.

$$B_{p,1}(\lambda, 1; x) = \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} x^{\lambda+j} \ln^{p-i+1} x, \quad (10)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$

More generally

$$B_{p,1}(\lambda, s; x) = \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^{i+1}}{j} B_{p,0}(\lambda+i+j, 1; x), \quad (11)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$

Proof. To prove equation (10), we have

$$\begin{aligned} \int_{\epsilon}^x t^{\lambda-1} \ln^p t \ln(1-t) dt &= - \sum_{j=1}^{\infty} \frac{1}{j} \int_{\epsilon}^x t^{\lambda+j-1} \ln^p t dt \\ &= \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} x^{\lambda+j} \ln^{p-i+1} x \\ &\quad - \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon \end{aligned} \quad (12)$$

on using equation (5).

Choosing a positive integer $J > -\lambda$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon &= \sum_{j=1}^J \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon \\ &+ \sum_{j=J+1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon, \quad (13) \end{aligned}$$

where

$$\sum_{j=J+1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon = O(\epsilon^{\lambda+J} \ln^{p-i+1} \epsilon).$$

It follows that

$$\text{N-lim}_{\epsilon \rightarrow 0} \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{j(p-i+1)! (\lambda+j)^i} \epsilon^{\lambda+j} \ln^{p-i+1} \epsilon = 0 \quad (14)$$

and it follows from equations (12) to (14) that

$$\begin{aligned} B_{p,1}(\lambda, 1; x) &= \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} \ln^p t \ln(1-t) dt \\ &= \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i p!}{(p-i+1)! j(\lambda+j)^i} x^{\lambda+j} \ln^{p-i+1} x, \end{aligned}$$

proving equation (10) for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

Next, we have

$$\begin{aligned} \int_{\epsilon}^x t^{\lambda-1} \ln^p t (1-t)^{s-1} \ln(1-t) dt \\ = - \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^i}{j} \int_{\epsilon}^x t^{\lambda+i+j-1} \ln^p t dt \end{aligned}$$

and equation (11) follows for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Corollary 2.1

$$B_{p,1}(\lambda, 1) = \sum_{j=1}^{\infty} \frac{(-1)^{p+1} p!}{j(\lambda+j)^{p+1}}, \quad (15)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

More generally

$$B_{p,1}(\lambda, s) = \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^{i+p+1} p!}{j(\lambda+i+j)^{p+1}}, \quad (16)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Proof. Equation (15) follows immediately from equations (4) and (10) for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

Equation (16) follows from equations (4), (8) and (11) for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Theorem 3.

$$B_{p,2}(\lambda, 1; x) = \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^{i+1} 2p! \phi(j)}{(p-i+1)!(j+1)(\lambda+j+1)^i} x^{\lambda+j+1} \ln^{p-i+1} x, \quad (17)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$, where $\phi(j) = \sum_{k=1}^j k^{-1}$, for $j = 1, 2, \dots$.

More generally

$$B_{p,2}(\lambda, s; x) = \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^i 2\phi(j)}{j+1} B_{p,0}(\lambda+i+j, 1; x), \quad (18)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Proof. We first of all note that

$$\begin{aligned} \ln^2(1-t) &= \sum_{j=1}^{\infty} \sum_{k=1}^j \frac{1}{k(j-k+1)} t^{j+1} \\ &= \sum_{j=1}^{\infty} \frac{2\phi(j)}{j+1} t^{j+1}. \end{aligned} \quad (19)$$

Then

$$\begin{aligned} \int_{\epsilon}^x t^{\lambda-1} \ln^p t \ln^2(1-t) dt &= \sum_{j=1}^{\infty} \frac{2\phi(j)}{j+1} \int_{\epsilon}^x t^{\lambda+j} \ln^p t dt \\ &= - \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i 2\phi(j) p!}{(p-i+1)!(j+1)(\lambda+j+1)^i} x^{\lambda+j+1} \ln^{p-i+1} x \\ &\quad + \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i 2\phi(j) p!}{(p-i+1)!(j+1)(\lambda+j+1)^i} \epsilon^{\lambda+j+1} \ln^{p-i+1} \epsilon \end{aligned} \quad (20)$$

on using equation (5).

It follows as in the proof of equation (14) that

$$\text{N-lim}_{\epsilon \rightarrow 0} \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^i 2\phi(j)p!}{(p-i+1)!(j+1)(\lambda+j+1)^i} \epsilon^{\lambda+j+1} \ln^{p-i+1} \epsilon = 0 \quad (21)$$

and so

$$\begin{aligned} B_{p,2}(\lambda, 1; x) &= \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-1} \ln^p t \ln^2(1-t) dt \\ &= \text{N-lim}_{\epsilon \rightarrow 0} \sum_{j=1}^{\infty} \frac{2\phi(j)}{j+1} \int_{\epsilon}^x t^{\lambda+j} \ln^p t dt \end{aligned}$$

on using equations (20) and (21). Equation (17) now follows for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

Next, we have

$$\begin{aligned} \int_{\epsilon}^x t^{\lambda-1} \ln^p t (1-t)^{s-1} \ln^2(1-t) dt \\ = \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^i 2\phi(j)}{j+1} \int_{\epsilon}^x t^{\lambda+i+j} \ln^p t dt \end{aligned}$$

and equation (18) follows for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Corollary 3.1

$$B_{p,2}(\lambda, 1) = \sum_{j=1}^{\infty} \frac{(-1)^j 2\phi(j)p!}{(j+1)(\lambda+j+1)^{p+1}}, \quad (22)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

More generally

$$B_{p,2}(\lambda, s) = \sum_{i=0}^{s-1} \sum_{j=1}^{\infty} \binom{s-1}{i} \frac{(-1)^{i+p} 2\phi(j)p!}{(j+1)(\lambda+i+j)^{p+1}}, \quad (23)$$

for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Proof. Equation (22) follows immediately from equations (4) and (17) for $\lambda \neq 0, -1, -2, \dots$ and $p = 0, 1, 2, \dots$.

Equation (23) follows from equations (4), (8) and (18) for $\lambda \neq 0, -1, -2, \dots$, $s = 1, 2, \dots$ and $p = 0, 1, 2, \dots$.

Theorem 4.

$$B_{p,1}(\lambda, 0; x) = \frac{\lambda-1}{2} B_{p,2}(\lambda-1, 1; x) + \frac{p}{2} B_{p-1,2}(\lambda-1, 1; x) - \frac{1}{2} x^{\lambda-1} \ln^p x \ln^2(1-x) \quad (24)$$

$$\begin{aligned} &= \frac{\lambda-1}{2} \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^{i+1} 2p! \phi(j)}{(p-i+1)!(j+1)(\lambda+j)^i} x^{\lambda+j} \ln^{p-i+1} x \\ &\quad + \frac{p}{2} \sum_{i=1}^p \sum_{j=1}^{\infty} \frac{(-1)^{i+1} 2(p-1)! \phi(j)}{(p-i)!(j+1)(\lambda+j)^i} x^{\lambda+j} \ln^{p-i} x \\ &\quad - \frac{1}{2} x^{\lambda-1} \ln^p x \ln^2(1-x), \end{aligned} \quad (25)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 1, 2, \dots$.

Proof. We have

$$\begin{aligned} \int_{\epsilon}^x t^{\lambda-2} \ln^p t \ln^2(1-t) dt &= \frac{1}{\lambda-1} \int_{\epsilon}^x \ln^p t \ln^2(1-t) dt^{\lambda-1} \\ &= \frac{1}{\lambda-1} x^{\lambda-1} \ln^p x \ln^2(1-x) \\ &\quad - \frac{1}{\lambda-1} \epsilon^{\lambda-1} \ln^p \epsilon \ln^2(1-\epsilon) \\ &\quad - \frac{1}{\lambda-1} \int_{\epsilon}^x t^{\lambda-1} \left[p t^{-1} \ln^{p-1} t \ln^2(1-t) \right. \\ &\quad \left. - 2 \ln^p t (1-t)^{-1} \ln(1-t) \right] dt, \end{aligned}$$

where it is clear that

$$\text{N-lim}_{\epsilon \rightarrow 0} \epsilon^{\lambda-1} \ln^p \epsilon \ln^2(1-\epsilon) = 0. \quad (26)$$

Using equation (15), we get

$$\begin{aligned} B_{p,2}(\lambda-1, 1; x) &= \text{N-lim}_{\epsilon \rightarrow 0} \int_{\epsilon}^x t^{\lambda-2} \ln^p t \ln^2(1-t) dt \\ &= \frac{1}{\lambda-1} x^{\lambda-1} \ln^p x \ln^2(1-x) - \frac{p}{\lambda-1} B_{p-1,2}(\lambda-1, 1; x) + \frac{2}{\lambda-1} B_{p,1}(\lambda, 0; x) \end{aligned}$$

and so

$$\begin{aligned}
 B_{p,1}(\lambda, 0; x) &= \frac{\lambda-1}{2} B_{p,2}(\lambda-1, 1; x) + \frac{p}{2} B_{p-1,2}(\lambda-1, 1; x) \\
 &\quad - \frac{1}{2} x^{\lambda-1} \ln^p x \ln^2(1-x) \\
 &= \frac{\lambda-1}{2} \sum_{i=1}^{p+1} \sum_{j=1}^{\infty} \frac{(-1)^{i+1} 2p! \phi(j)}{(p-i+1)!(j+1)(\lambda+j)^i} x^{\lambda+j} \ln^{p-i+1} x \\
 &\quad + \frac{p}{2} \sum_{i=1}^p \sum_{j=1}^{\infty} \frac{(-1)^{i+1} 2(p-1)! \phi(j)}{(p-i)!(j+1)(\lambda+j)^i} x^{\lambda+j} \ln^{p-i} x \\
 &\quad - \frac{1}{2} x^{\lambda-1} \ln^p x \ln^2(1-x),
 \end{aligned}$$

proving equations (24) and (25).

Corollary 4.1

$$B_{p,1}(\lambda, 0) = \frac{\lambda-1}{2} B_{p,2}(\lambda-1, 1) + \frac{p}{2} B_{p-1,2}(\lambda-1, 1) \quad (27)$$

$$= \frac{\lambda-1}{2} \sum_{j=1}^{\infty} \frac{(-1)^p 2\phi(j)p!}{(j+1)(\lambda+j)^{p+1}} + \frac{p}{2} \sum_{j=1}^{\infty} \frac{(-1)^{p-1} 2\phi(j)(p-1)!}{(j+1)(\lambda+j)^p}, \quad (28)$$

for $\lambda \neq 0, -1, -2, \dots$ and $p = 1, 2, \dots$.

Proof. We have

$$\begin{aligned}
 B_{p,1}(\lambda, 0) &= \lim_{\epsilon \rightarrow 0} \left[\frac{\lambda-1}{2} B_{p,2}(\lambda-1, 1; 1-\epsilon) + \frac{p}{2} B_{p-1,2}(\lambda-1, 1; 1-\epsilon) \right. \\
 &\quad \left. - \frac{1}{2} (1-\epsilon)^{\lambda-1} \ln^p(1-\epsilon) \ln^2 \epsilon \right] \\
 &= \frac{\lambda-1}{2} B_{p,2}(\lambda-1, 1) + \frac{p}{2} B_{p-1,2}(\lambda-1, 1) \\
 &= \frac{\lambda-1}{2} \sum_{j=1}^{\infty} \frac{(-1)^p 2\phi(j)p!}{(j+1)(\lambda+j)^{p+1}} + \frac{p}{2} \sum_{j=1}^{\infty} \frac{(-1)^{p-1} 2\phi(j)(p-1)!}{(j+1)(\lambda+j)^p},
 \end{aligned}$$

on using equations (4) and (22), proving equations (27) and (28) for $p = 1, 2, \dots$.

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