ITERATIVE METHODS FOR SOLVING SYSTEM OF FUZZY SYLVESTER EQUATIONS

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Abstract: In this paper, we present Jacobi and Adomian decomposition iterative methods for finding the approximate solution of system of fuzzy Sylvester equations (SFSE), $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. It is shown that for a SFSE the Adomian decomposition method is equivalent to the Jacobi iterative method. Finally, the proposed iterative methods are illustrated by solving one example.

AMS Subject Classification: 15B15

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1. Introduction

Finding solutions of systems of linear equations plays a crucial role in almost every field of science and engineering. In many applications, some of the system parameters are represented by fuzzy numbers rather than by crisp numbers. Therefore, it is important to develop mathematical models and numerical procedures to general system of fuzzy linear equations (SFLE). In [11] it is considered a general model for solving a SFLE whose coefficient matrix is crisp and its right hand side is an arbitrary fuzzy vector. The authors stated some conditions for the existence of a unique fuzzy solution to SFLE by the embedding method and converting the original system to a crisp linear system

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of equations. Many authors studied SFLE. Numerical methods for SFLE were proposed by Allahviranloo [16, 17]. Dehghan and Hashemi [10] applied several iterative methods for solving SFLE. Wang and Zheng [8] studied some block iterative methods to solve SFLE. The Sylvester matrix equation has the form

$$AX + XB = C, \quad (1)$$

where A is an $n \times n$ crisp matrix, B is an $m \times m$ crisp matrix, C is an $n \times m$ fuzzy matrix and X is an $n \times m$ unknown matrix. This matrix equation plays an important role in control theory, signal processing, filtering, model reduction, image restoration, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, and block-diagonalization of matrices. See for example, Benner [12, 13], Datta and Datta [2], Hyland and Bernstein [4], Laub et al. [1]. Standard solution methods for Sylvester equations of the form (1) are the Bartels-Stewart method in [14] and the Hessenberg Schur method in [6]. These methods are based on transforming the coefficient matrices to Schur or Hessenberg form and then solving the corresponding linear system of equations directly by a backward substitution process. Therefore, these methods are classed as direct methods.

In this paper, we discuss some cases of system of fuzzy Sylvester equations (1), where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. The paper is organized as follows:

In Section 2, we introduce some basic definitions and results on fuzzy numbers and system of fuzzy Sylvester equations. In Section 3, we propose Jacobi and Adomian decomposition iterative methods for finding the approximate solution system of fuzzy Sylvester equations. In Section 4, we illustrate the proposed iterative methods by solving one example. Conclusions are drawn in Section 5.

2. Preliminaries

Definition 1. A fuzzy number U is an ordered pair of functions $(\underline{U}(\alpha), \overline{U}(\alpha))$, $0 \leq \alpha \leq 1$, which satisfies the following conditions:

1. $\underline{U}(\alpha)$ is a continuous, monotonically increasing function on $[0, 1]$.
2. $\overline{U}(\alpha)$ is a continuous, monotonically decreasing function on $[0, 1]$.
3. $\underline{U}(\alpha) \leq \overline{U}(\alpha)$ on $[0, 1]$.

Definition 2. For arbitrary fuzzy numbers $U = (\underline{U}(\alpha), \overline{U}(\alpha))$, $V = (\underline{V}(\alpha), \overline{V}(\alpha))$ and a scalar k , we define addition, subtraction and scalar multiplication by k as:

1. $(U + V)(\alpha) = (\underline{U}(\alpha) + \underline{V}(\alpha), \overline{U}(\alpha) + \overline{V}(\alpha))$,
 2. $(U - V)(\alpha) = (\underline{U}(\alpha) - \overline{V}(\alpha), \overline{U}(\alpha) - \underline{V}(\alpha))$,
 3. $(kU)(\alpha) = \begin{cases} (k\underline{U}(\alpha), k\overline{U}(\alpha)) & k \geq 0, \\ (k\overline{U}(\alpha), k\underline{U}(\alpha)) & k < 0, \end{cases}$
 4. $U = V$ iff $\underline{U}(\alpha) = \underline{V}(\alpha), \overline{U}(\alpha) = \overline{V}(\alpha)$.
- (2)

Definition 3. The system of linear equations

$$AX + XB = C, \quad (3)$$

is called system of fuzzy Sylvester equations if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times m$ crisp matrices and $C = (c_{ij})$ is an $m \times m$ fuzzy matrix.

Definition 4. The system of linear equations $AX + XB = C$ is called system of fully fuzzy Sylvester equations if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are $m \times m$ fuzzy matrices.

Definition 5. A fuzzy matrix $X(\alpha) = (x_{ij}(\alpha)) = (\underline{x}_{ij}(\alpha), \overline{x}_{ij}(\alpha))$, $1 \leq i, j \leq m$, $0 \leq \alpha \leq 1$, is called the solution of fuzzy Sylvester equations, if

$$\begin{aligned} \left(\sum_{k=1}^m a_{ik} x_{kj} + \sum_{k=1}^m x_{ik} b_{kj} \right) (\alpha) &= \sum_{k=1}^m a_{ik} x_{kj}(\alpha) + \sum_{k=1}^m x_{ik} b_{kj}(\alpha) \\ &= \sum_{k=1}^m \underline{a_{ik} x_{kj}}(\alpha) + \sum_{k=1}^m \underline{x_{ik} b_{kj}}(\alpha) = \underline{c_{ij}}(\alpha), \\ \left(\sum_{k=1}^m a_{ik} x_{kj} + \sum_{k=1}^m x_{ik} b_{kj} \right) (\alpha) &= \overline{\sum_{k=1}^m a_{ik} x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik} b_{kj}(\alpha)} \\ &= \sum_{k=1}^m \overline{a_{ik} x_{kj}}(\alpha) + \sum_{k=1}^m \overline{x_{ik} b_{kj}}(\alpha) = \overline{c_{ij}}(\alpha). \end{aligned} \quad (4)$$

In particular, if $b_{kj} \geq 0, 1 \leq k \leq m, 1 \leq j \leq m$ and $a_{ik} \geq 0, 1 \leq i \leq m, 1 \leq k \leq m$, we easily get

$$\begin{aligned}
 \overline{\left(\sum_{k=1}^m a_{ik} x_{kj} + \sum_{k=1}^m x_{ik} b_{kj} \right)}(\alpha) &= \overline{\sum_{k=1}^m a_{ik} x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik} b_{kj}(\alpha)} \\
 &= \sum_{k=1}^m a_{ik} \underline{x}_{kj}(\alpha) + \sum_{k=1}^m \underline{x}_{ik}(\alpha) b_{kj} = \underline{c}_{ij}(\alpha), \\
 \overline{\left(\sum_{k=1}^m a_{ik} x_{kj} + \sum_{k=1}^m x_{ik} b_{kj} \right)}(\alpha) &= \overline{\sum_{k=1}^m a_{ik} x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik} b_{kj}(\alpha)} \\
 &= \sum_{k=1}^m a_{ik} \overline{x}_{kj}(\alpha) + \sum_{k=1}^m \overline{x}_{ik}(\alpha) b_{kj} = \overline{c}_{ij}(\alpha). \tag{5}
 \end{aligned}$$

Consider the ij -th equation of the system (3):

$$\sum_{k=1}^m a_{ik}(\underline{x}_{kj}, \overline{x}_{kj}) + \sum_{k=1}^m (\underline{x}_{ik}, \overline{x}_{ik}) b_{kj} = (\underline{c}_{ij}, \overline{c}_{ij}), 1 \leq i, j \leq m. \tag{6}$$

From (6) we have two $m^2 \times m^2$ crisp linear systems for all $1 \leq i, j \leq m$ that there can be extended to an $2m^2 \times 2m^2$ crisp linear system as follows:

$$SZ + ZT = Y, \tag{7}$$

where

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} + \begin{pmatrix} \underline{X} & \overline{X} \\ \overline{X} & \underline{X} \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix} = \begin{pmatrix} \underline{C} & \overline{C} \\ \overline{C} & \underline{C} \end{pmatrix}. \tag{8}$$

Thus the system of fuzzy linear equations (3) is extended to a system of crisp linear equations (8), where $A = S_1 + S_2, S_1 \geq 0, S_2 \leq 0$ and $B = T_1 + T_2, T_1 \geq 0, T_2 \leq 0$.

The system of equations (8) can be written as follows:

$$\begin{cases} S_1 \underline{X} + S_2 \overline{X} + \underline{X} T_1 + \overline{X} T_2 = \underline{C}, \\ S_2 \underline{X} + S_1 \overline{X} + \underline{X} T_2 + \overline{X} T_1 = \overline{C}. \end{cases} \tag{9}$$

If $A = S_1 - S_2, S_1 \geq 0, S_2 \geq 0$ and $B = T_1 - T_2, T_1 \geq 0, T_2 \geq 0$, the system of equations (8) becomes:

$$\begin{cases} S_1 \underline{X} - S_2 \overline{X} + \underline{X} T_1 - \overline{X} T_2 = \underline{C}, \\ -S_2 \underline{X} + S_1 \overline{X} + \underline{X} T_1 - \overline{X} T_2 = \overline{C}. \end{cases} \tag{10}$$

Theorem 6. *The matrices $A = (a_{ij})$ and $B = (b_{ij})$, $1 \leq i, j \leq m$, in the system of equations (3) are both positive definite iff the matrices S and T in the system of equations (7) are both positive definite.*

Proof. See [15]. □

Theorem 7. *The matrices $A = (a_{ij})$ and $B = (b_{ij})$, $1 \leq i, j \leq m$, with $a_{ii} > 0$ and $b_{ii} > 0$ as in the system of equations (3) are both strictly diagonally dominant iff the matrices S and T in the system of equations (7) are both strictly diagonally dominant.*

Proof. See [9]. □

3. The Jacobi and Adomian Decomposition Iterative Methods

In this section, we use the Jacobi and Adomian decomposition iterative methods to solve the system of equations (3). First we present the Jacobi iterative method.

Without loss of generality, suppose that in system of equations (8), $s_{ii} > 0$ and $t_{ii} > 0$ for all $1 \leq i \leq 2m$. Let $S = L + D + U$, where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 \\ S_2 & L_1 \end{pmatrix}, U = \begin{pmatrix} U_1 & S_2 \\ 0 & U_1 \end{pmatrix},$$

and also $T = L' + D' + U'$, where

$$D' = \begin{pmatrix} D'_1 & 0 \\ 0 & D'_1 \end{pmatrix}, L' = \begin{pmatrix} L'_1 & 0 \\ T_2 & L'_1 \end{pmatrix}, U' = \begin{pmatrix} U'_1 & T_2 \\ 0 & U'_1 \end{pmatrix},$$

$(D_1)_{ii} = s_{ii} > 0, 1 \leq i \leq m$ and $(D'_1)_{ii} = t_{ii} > 0, 1 \leq i \leq m$ and suppose $S_1 = L_1 + D_1 + U_1$ and $T_1 = L'_1 + D'_1 + U'_1$. In the Jacobi method, from the structure of $SZ + ZT = Y$ we have

$$\begin{aligned} & \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \underline{X} & \overline{X} \end{pmatrix} + \begin{pmatrix} L_1 + U_1 & S_2 \\ S_2 & L_1 + U_1 \end{pmatrix} \begin{pmatrix} \underline{X} & \overline{X} \\ \underline{X} & \overline{X} \end{pmatrix} \\ & + \begin{pmatrix} \underline{X} & \overline{X} \\ \underline{X} & \overline{X} \end{pmatrix} \begin{pmatrix} D'_1 & 0 \\ 0 & D'_1 \end{pmatrix} + \begin{pmatrix} \underline{X} & \overline{X} \\ \underline{X} & \overline{X} \end{pmatrix} \begin{pmatrix} L'_1 + U'_1 & T_2 \\ T_2 & L'_1 + U'_1 \end{pmatrix} = \begin{pmatrix} \underline{C} & \overline{C} \\ \underline{C} & \overline{C} \end{pmatrix}. \quad (11) \end{aligned}$$

Then

$$\begin{cases} D_1 \underline{X} + \underline{X} D_1' = \underline{C} - (L_1 + U_1) \underline{X} - S_2 \overline{X} - \underline{X} (L_1' + U_1') - \overline{X} T_2, \\ D_1 \overline{X} + \overline{X} D_1' = \overline{C} - (L_1 + U_1) \overline{X} - S_2 \underline{X} - \underline{X} T_2 - \overline{X} (L_1' + U_1'). \end{cases} \quad (12)$$

The elements of $X^{(k+1)}(\alpha) = (\underline{X}^{(k+1)}(\alpha), \overline{X}^{(k+1)}(\alpha))$, $0 \leq \alpha \leq 1$ are

$$\begin{aligned} \underline{x}_{ij}^{(k+1)} &= \frac{1}{s_{ii} + t_{jj}} \left[\underline{c}_{ij}(\alpha) - \sum_{i \neq l, l=1}^m s_{il} \underline{x}_{lj}^{(k)}(\alpha) - \sum_{l=1}^m s_{i,m+l} \overline{x}_{lj}^{(k)}(\alpha) \right. \\ &\quad \left. - \sum_{j \neq l, l=1}^m \underline{x}_{il}^{(k)}(\alpha) t_{lj} - \sum_{l=1}^m \overline{x}_{il}^{(k)}(\alpha) t_{l,m+j} \right], \\ \overline{x}_{ij}^{(k+1)} &= \frac{1}{s_{ii} + t_{jj}} \left[\overline{c}_{ij}(\alpha) - \sum_{i \neq l, l=1}^m s_{il} \overline{x}_{lj}^{(k)}(\alpha) - \sum_{l=1}^m s_{i,m+l} \underline{x}_{lj}^{(k)}(\alpha) \right. \\ &\quad \left. - \sum_{j \neq l, l=1}^m \overline{x}_{il}^{(k)}(\alpha) t_{lj} - \sum_{l=1}^m \underline{x}_{il}^{(k)}(\alpha) t_{l,m+j} \right], \\ k &= 0, 1, \dots, \quad 1 \leq i, j \leq m. \end{aligned} \quad (13)$$

The stopping criterion with tolerance $\varepsilon > 0$ is

$$\frac{\|\overline{X}^{(k+1)} - \overline{X}^{(k)}\|}{\|\overline{X}^{(k+1)}\|} < \varepsilon, \quad \frac{\|\underline{X}^{(k+1)} - \underline{X}^{(k)}\|}{\|\underline{X}^{(k+1)}\|} < \varepsilon, \quad k = 0, 1, \dots$$

Now we present the Adomian decomposition method. Considering (9) and (12) for any $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} \underline{x}_{ij}(\alpha) &= \frac{\underline{c}_{ij}(\alpha)}{s_{ii} + t_{jj}} - \frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \underline{x}_{lj}(\alpha) + \sum_{l=1}^m s_{i,m+l} \overline{x}_{lj}(\alpha) \right. \\ &\quad \left. + \sum_{j \neq l, l=1}^m \underline{x}_{il}(\alpha) t_{lj} + \sum_{l=1}^m \overline{x}_{il}(\alpha) t_{l,m+j} \right], \\ \overline{x}_{ij}(\alpha) &= \frac{\overline{c}_{ij}(\alpha)}{s_{ii} + t_{jj}} - \frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \overline{x}_{lj}(\alpha) + \sum_{l=1}^m s_{i,m+l} \underline{x}_{lj}(\alpha) \right. \\ &\quad \left. + \sum_{j \neq l, l=1}^m \overline{x}_{il}(\alpha) t_{lj} + \sum_{l=1}^m \underline{x}_{il}(\alpha) t_{l,m+j} \right], \quad 1 \leq i, j \leq m. \end{aligned} \quad (14)$$

The canonical form of the Adomian equation can be written as:

$$\begin{aligned}\underline{x}_{ij} &= \underline{d}_{ij} + \underline{g}_{ij}(\underline{x}_{lj}; i \neq l, \bar{x}_{lj}, \underline{x}_{il}; j \neq l, \bar{x}_{il}), \\ \bar{x}_{ij} &= \bar{d}_{ij} + \bar{g}_{ij}(\bar{x}_{lj}; i \neq l, \underline{x}_{lj}, \bar{x}_{il}; j \neq l, \underline{x}_{il}),\end{aligned}\quad (15)$$

where $\underline{g}_{ij}$ and $\bar{g}_{ij}$ are linear functions and $\underline{d}_{ij}, \bar{d}_{ij}$ are constants.

Let $\underline{x}_{ij} = \sum_{r=0}^{\infty} \underline{x}_{i,j,r}$, $\bar{x}_{ij} = \sum_{r=0}^{\infty} \bar{x}_{i,j,r}$, and

$$\begin{aligned}\underline{g}_{ij}(\underline{x}_{lj}; i \neq l, \bar{x}_{lj}, \underline{x}_{il}; j \neq l, \bar{x}_{il}) &= \sum_{r=0}^{\infty} \underline{A}_{i,j,r}, \\ \bar{g}_{ij}(\bar{x}_{lj}; i \neq l, \underline{x}_{lj}, \bar{x}_{il}; j \neq l, \underline{x}_{il}) &= \sum_{r=0}^{\infty} \bar{A}_{i,j,r},\end{aligned}$$

where $\underline{A}_{i,j,r}, \bar{A}_{i,j,r}$, ($r = 0, 1, \dots$) with components of $\underline{g}_{ij}, \bar{g}_{ij}$ in (15) are called Adomian polynomials and can be written as:

$$\begin{aligned}\underline{A}_{i,j,r} &= \frac{1}{r!} \left[\frac{d^r}{d\lambda^r} \underline{g}_{ij} \left(\sum_{k=0}^{\infty} \lambda^k \underline{x}_{1,1,k}, \dots, \sum_{k=0}^{\infty} \lambda^k \underline{x}_{m,m,k}, \right. \right. \\ &\quad \left. \left. \sum_{k=0}^{\infty} \lambda^k \bar{x}_{1,1,k}, \dots, \sum_{k=0}^{\infty} \lambda^k \bar{x}_{m,m,k} \right) \right]_{\lambda=0}, \\ \bar{A}_{i,j,r} &= \frac{1}{r!} \left[\frac{d^r}{d\lambda^r} \bar{g}_{ij} \left(\sum_{k=0}^{\infty} \lambda^k \bar{x}_{1,1,k}, \dots, \sum_{k=0}^{\infty} \lambda^k \bar{x}_{m,m,k}, \right. \right. \\ &\quad \left. \left. \sum_{k=0}^{\infty} \lambda^k \underline{x}_{1,1,k}, \dots, \sum_{k=0}^{\infty} \lambda^k \underline{x}_{m,m,k} \right) \right]_{\lambda=0}.\end{aligned}$$

Hence, equation (15) can be written as:

$$\begin{cases} \sum_{r=0}^{\infty} \underline{x}_{i,j,r} = \underline{d}_{ij} + \sum_{r=0}^{\infty} \underline{A}_{i,j,r}, \\ \sum_{r=0}^{\infty} \bar{x}_{i,j,r} = \bar{d}_{ij} + \sum_{r=0}^{\infty} \bar{A}_{i,j,r}, \\ 1 \leq i, j \leq m, r = 0, 1, \dots \end{cases} \quad (16)$$

From (16), we define:

$$\left\{ \begin{array}{l} \underline{x}_{i,j,0} = \underline{d}_{ij}, \\ \underline{x}_{i,j,r+1} = \underline{A}_{i,j,r}, \quad \underline{\phi}_{i,j,k} = \sum_{r=0}^{k-1} \underline{x}_{i,j,r}, \quad 1 \leq i, j \leq m, r = 0, 1, \dots \end{array} \right. \quad (17)$$

$$\left\{ \begin{array}{l} \bar{x}_{i,j,0} = \bar{d}_{ij}, \\ \bar{x}_{i,j,r+1} = \bar{A}_{i,j,r}, \quad \bar{\phi}_{i,j,k} = \sum_{r=0}^{k-1} \bar{x}_{i,j,r}, \quad 1 \leq i, j \leq m, r = 0, 1, \dots \end{array} \right. \quad (18)$$

Thus $\lim_{k \rightarrow \infty} \underline{\phi}_{i,j,r} = \underline{x}_{ij}$, $\lim_{k \rightarrow \infty} \bar{\phi}_{i,j,k} = \bar{x}_{ij}$.

To determine the Adomian polynomials of order n , we write:

$$\underline{x}_{i,j,\lambda} = \sum_{r=0}^{\infty} \lambda^r \underline{x}_{i,j,r}, \quad \bar{x}_{i,j,\lambda} = \sum_{r=0}^{\infty} \lambda^r \bar{x}_{i,j,r}, \quad (19)$$

$$\underline{g}_{i,j,\lambda}(\underline{x}_{lj}; i \neq l, \bar{x}_{lj}, \underline{x}_{il}; j \neq l, \bar{x}_{il}) = \sum_{r=0}^{\infty} \lambda^r \underline{A}_{i,j,r},$$

$$\bar{g}_{i,j,\lambda}(\bar{x}_{lj}; i \neq l, \underline{x}_{lj}, \bar{x}_{il}; j \neq l, \underline{x}_{il}) = \sum_{r=0}^{\infty} \lambda^r \bar{A}_{i,j,r}, \quad 1 \leq i, j \leq m, \quad (20)$$

where λ is a parameter. From (19) and (20) we obtain:

$$\begin{aligned} \underline{A}_{i,j,r} &= \frac{1}{r!} \left[\frac{d^r}{d\lambda^r} \underline{g}_{i,j,\lambda}(\underline{x}_{lj}; i \neq l, \bar{x}_{lj}, \underline{x}_{il}; j \neq l, \bar{x}_{il}) \right]_{\lambda=0} \\ &= -\frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \underline{x}_{l,j,r} + \sum_{l=1}^m s_{i,n+l} \bar{x}_{l,j,r} + \sum_{j \neq l, l=1}^m \underline{x}_{l,j,r} t_{lj} + \sum_{l=1}^m t_{l,n+j} \bar{x}_{l,j,r} \right], \\ \bar{A}_{i,j,r} &= \frac{1}{r!} \left[\frac{d^r}{d\lambda^r} \bar{g}_{i,j,\lambda}(\bar{x}_{lj}; i \neq l, \underline{x}_{lj}, \bar{x}_{il}; j \neq l, \underline{x}_{il}) \right]_{\lambda=0} \\ &= -\frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \bar{x}_{l,j,r} + \sum_{l=1}^m s_{i,n+l} \underline{x}_{l,j,r} + \sum_{j \neq l, l=1}^m \bar{x}_{l,j,r} t_{lj} + \sum_{l=1}^m t_{l,n+j} \underline{x}_{l,j,r} \right], \\ &\quad 1 \leq i, j \leq m, r = 0, 1, \dots \end{aligned} \quad (21)$$

From (16), (17), (18) and (21) the solutions of system of fuzzy linear equations are as follows: For any $0 \leq \alpha \leq 1$,

$$\left\{ \begin{array}{l} \underline{x}_{i,j,0}(\alpha) = \underline{d}_{ij}(\alpha), \\ \underline{x}_{i,j,m+1}(\alpha) = -\frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \underline{x}_{l,j,r}(\alpha) + \sum_{l=1}^m s_{i,n+l} \bar{x}_{l,j,r}(\alpha) \right. \\ \quad \left. + \sum_{j \neq l, l=1}^m \underline{x}_{l,j,r}(\alpha) t_{lj} + \sum_{l=1}^m \bar{x}_{l,j,r}(\alpha) t_{l,n+j} \right], \\ r = 0, 1, \dots, \quad 1 \leq i, j \leq m; \end{array} \right. \quad (22)$$

$$\left\{ \begin{array}{l} \bar{x}_{i,j,0}(\alpha) = \bar{d}_{ij}(\alpha), \\ \bar{x}_{i,j,m+1}(\alpha) = -\frac{1}{s_{ii} + t_{jj}} \left[\sum_{i \neq l, l=1}^m s_{il} \bar{x}_{l,j,r}(\alpha) + \sum_{l=1}^m s_{i,n+l} \underline{x}_{l,j,r}(\alpha) \right. \\ \quad \left. + \sum_{j \neq l, l=1}^m \bar{x}_{l,j,r}(\alpha) t_{lj} + \sum_{l=1}^m \underline{x}_{l,j,r}(\alpha) t_{l,n+j} \right], \\ r = 0, 1, \dots, \quad 1 \leq i, j \leq m. \end{array} \right. \quad (23)$$

Note that (22) and (23) are the same as (13) and the iteration matrix of the Adomian decomposition method is the same as the iteration matrix of the Jacobi iterative method. The initial values for the Adomian decomposition method are:

$$\underline{x}_{ij}^0 = \frac{\underline{c}_{ij}}{s_{ii} + t_{jj}}, \bar{x}_{ij}^0 = \frac{\bar{c}_{ij}}{s_{ii} + t_{jj}}, 1 \leq i, j \leq m,$$

while for the Jacobi iterative method they are usually $\underline{X}^0 = 0$ and $\bar{X}^0 = 0$. So in this case the Adomian decomposition method does not give a new solution, but one iterate less than the Jacobi iteration method is used.

4. Numerical Examples

In this section we illustrate and compare the methods from Section 3 by solving an example.

Example 8. Consider the system of fuzzy Sylvester equations $AX + XB = C$, where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} (0, 1, 2) & (1, 2, 3) \\ (1, 2, 3) & (0, 1, 2) \end{pmatrix}.$$

We extend the A matrix to the S matrix and the B matrix to the T matrix, as follows:

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

The exact solution is given by

$$\begin{aligned} x_{11} &= (0.6935 + 0.2857\alpha, 1.2649 - 0.2857\alpha), \\ x_{12} &= (0.4018 + 0.2857\alpha, 0.9732 - 0.2857\alpha), \\ x_{21} &= (0.1280 + 0.1429\alpha, 0.4137 - 0.1429\alpha), \\ x_{22} &= (-0.0804 + 0.1429\alpha, 0.2054 - 0.1429\alpha). \end{aligned}$$

By equations (17), (18), the Adomian decomposition the approximate solution with 6 iterates and $\varepsilon = 10^{-2}$ and by using equation (13) are the same as the Jacobi approximate solution with 7 iterates with $\varepsilon = 10^{-2}$:

$$\begin{aligned} x_{11} &= (0.6946 + 0.2866\alpha, 1.2678 - 0.2866\alpha), \\ x_{12} &= (0.4018 + 0.2857\alpha, 0.9732 - 0.2857\alpha), \\ x_{21} &= (0.1272 + 0.1433\alpha, 0.4138 - 0.1432\alpha), \\ x_{22} &= (-0.0804 + 0.1429\alpha, 0.2053 - 0.1429\alpha). \end{aligned}$$

The exact and approximate solutions are taken for $\alpha = 0, \alpha = 0.5, \alpha = 1$ with $\varepsilon = 10^{-2}$ in the following tables.

$\alpha = 0.0$	Exact solution	Jacobi method	Adomian decomposition
$(\underline{x}_{11}, \bar{x}_{11})$	(0.6935, 1.2649)	(0.6946, 1.2678)	(0.6946, 1.2678)
$(\underline{x}_{12}, \bar{x}_{12})$	(0.4018, 0.9732)	(0.4018, 0.9732)	(0.4018, 0.9732)
$(\underline{x}_{21}, \bar{x}_{21})$	(0.1280, 0.4137)	(0.1272, 0.4138)	(0.1272, 0.4138)
$(\underline{x}_{22}, \bar{x}_{22})$	(-0.0804, 0.2054)	(-0.0804, 0.2053)	(-0.0804, 0.2053)

$\alpha = 0.5$	Exact solution	Jacobi method	Adomian decomposition
$(\underline{x}_{11}, \bar{x}_{11})$	(0.8364, 1.1221)	(0.8379, 1.1245)	(0.8379, 1.1245)
$(\underline{x}_{12}, \bar{x}_{12})$	(0.5446, 0.8303)	(0.5446, 0.8303)	(0.5446, 0.8303)
$(\underline{x}_{21}, \bar{x}_{21})$	(0.1995, 0.3423)	(0.1989, 0.3422)	(0.1989, 0.3422)
$(\underline{x}_{22}, \bar{x}_{22})$	(-0.0089, 0.1340)	(-0.0089, 0.1339)	(-0.0089, 0.1339)

$\alpha = 1.0$	Exact solution	Jacobi method	Adomian decomposition
$(\underline{x}_{11}, \overline{x}_{11})$	(0.9792, 0.9792)	(0.9812, 0.9812)	(0.9812, 0.9812)
$(\underline{x}_{12}, \overline{x}_{12})$	(0.6875, 0.6875)	(0.6875, 0.6875)	(0.6875, 0.6875)
$(\underline{x}_{21}, \overline{x}_{21})$	(0.2709, 0.2709)	(0.2705, 0.2706)	(0.2705, 0.2706)
$(\underline{x}_{22}, \overline{x}_{22})$	(0.0625, 0.0625)	(0.0625, 0.0624)	(0.0625, 0.0624)

5. Conclusion

In this paper, we apply Jacobi and Adomian decomposition iterative methods for finding the approximate solution of system of fuzzy Sylvester equations of the form $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. We show that the Adomian method is equal to the Jacobi method.

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