THE KRONECKER PRODUCT FOR SOLVING SYSTEM OF FUZZY SYLVESTER EQUATIONS

Azim Rivaz^{1 §}, Fatemeh Salary Pour Sharif Abad²

¹Department of Mathematics, Shahid Bahonar
University of Kerman
Kerman, 76169-14111, IRAN

Abstract: In this paper, we present a method for finding solution of system of fuzzy Sylvester equations (SFSE), $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. Finally, the method is illustrated by solving some numerical examples.

AMS Subject Classification: 15B15

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1. Introduction

The systems of linear equations and their solutions play a crucial role in almost every field of science and engineering. In many applications, some of the system parameters are represented by fuzzy numbers rather than crisp numbers. Therefore, it is important to develop mathematical models and numerical procedures to general system of fuzzy linear equations (SFLE). In [2] it is considered a general model for solving a SFLE whose coefficient matrix is crisp and its right hand side is an arbitrary fuzzy vector. The authors stated some conditions for the existence of a unique fuzzy solution to SFLE by embedding method and

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[§]Correspondence author

converting the original system to a crisp linear system of equations. Note that many authors have studied SFLE. Numerical methods for SFLE were proposed by Allahviranloo [14, 15]. Dehghan and Hashemi [3] applied several iterative methods for solving SFLE. Wang and Zheng [5] studied some block iterative methods to solve SFLE. The so-called Sylvester matrix equation has the form

$$AX + XB = C, \quad (1)$$

where A is an $n \times n$ crisp matrix, B is an $m \times m$ crisp matrix, C is an $n \times m$ fuzzy matrix and X is an $n \times m$ unknown matrix. This matrix equation plays an important role in control theory, signal processing, filtering, model reduction, image restoration, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, and block-diagonalization of matrices. See for example, Benner [12, 13], Datta and Datta [4], Hyland and Bernstein [7], Laub et al. [1]. Standard solution methods for Sylvester equations of the form (1) are the Bartels-Stewart method in [11] and the Hessenberg-Schur method in [6]. These methods are based on the transforming the coefficient matrices to Schur or Hessenberg form and then solving the corresponding linear system of equations directly by a backward substitution process. Therefore, these methods are classed as direct methods.

In this paper, we consider some cases of system of fuzzy Sylvester equations (1) where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. The paper is organized as follows:

In Section 2, we introduce some basic definitions and results on fuzzy numbers and system fuzzy Sylvester equations. In Section 3, we propose a method for finding solution of fuzzy Sylvester equations. In Section 4, we illustrate the proposed method by solving some numerical examples. Conclusions are drawn in Section 5.

2. Preliminaries

Definition 1. A fuzzy number U is an ordered pair of functions $(\underline{U}(\alpha), \overline{U}(\alpha)), 0 \leq \alpha \leq 1$, which satisfies the following conditions:

1. $\underline{U}(\alpha)$ is a continuous, monotonically increasing function on $[0, 1]$.
2. $\overline{U}(\alpha)$ is a continuous, monotonically decreasing function on $[0, 1]$.
3. $\underline{U}(\alpha) \leq \overline{U}(\alpha)$ on $[0, 1]$.

Definition 2. For arbitrary fuzzy numbers $U = (\underline{U}(\alpha), \overline{U}(\alpha)), V =$

$(\underline{V}(\alpha), \overline{V}(\alpha))$ and a scalar k , we define addition, subtraction and scalar multiplication by k as:

1. $(U + V)(\alpha) = (\underline{U}(\alpha) + \underline{V}(\alpha), \overline{U}(\alpha) + \overline{V}(\alpha)),$
2. $(U - V)(\alpha) = (\underline{U}(\alpha) - \underline{V}(\alpha), \overline{U}(\alpha) - \overline{V}(\alpha)),$
3. $(kU)(\alpha) = \begin{cases} (k\underline{U}(\alpha), k\overline{U}(\alpha)) & k \geq 0 \\ (k\overline{U}(\alpha), k\underline{U}(\alpha)) & k < 0, \end{cases}$
4. $U = V \iff \underline{U}(\alpha) = \underline{V}(\alpha), \overline{U}(\alpha) = \overline{V}(\alpha).$

(2)

Definition 3. The linear matrix equation $AXB = C$ is termed fuzzy linear matrix equations (FLME), if the right coefficient matrix $B = (b_{ij}), 1 \leq i \leq r, 1 \leq j \leq l$ and the left coefficient matrix $A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$ are two crisp matrices and the right-hand-side matrix $C = (c_{ij}), 1 \leq i \leq m, 1 \leq j \leq l$ is a fuzzy matrix.

The ij -th equation of this system is:

$$\sum_{t=1}^r \sum_{k=1}^n a_{ik} x_{kt} b_{tj} = c_{ij}, 1 \leq i \leq m, 1 \leq j \leq l. \quad (3)$$

Definition 4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $r \times l$ matrices, respectively. Then the Kronecker product $(A \otimes B)$ is an $mr \times nl$ matrix, so that each partitioned matrix is like $(A \otimes B)_{ij} = a_{ij}B, i = 1, \dots, m, j = 1, \dots, n.$

The formal rules for operating with Kronecker products are as follows:

1. $0 \otimes A = A \otimes 0 = 0,$
2. $(A_1 + A_2) \otimes B = (A_1 \otimes B) + (A_2 \otimes B),$
3. $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2,$
4. $\alpha A \otimes \beta B = \alpha\beta(A \otimes B),$
5. $A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2).$

(4)

We use the Kronecker product for solving of fuzzy linear matrix equations (FLME), $AXB = C$ where A is an $m \times n$ matrix and B is an $r \times l$ matrix and we can write $AXB = C$ in the form of linear equations $(A \otimes B^t)x = c$ where $(A \otimes B^t)$ is the Kronecker product of A and B , an $ml \times nr$ matrix, x is an nr vector obtained by writing the columns of X one below another (i.e. $(i+1)$ -st following the i -th), and c is an ml vector similarly obtained from C .

3. Fuzzy Sylvester Equations

In this section, we investigate system of fuzzy Sylvester equations and propose a procedure for solving these equations.

Definition 5. The system of linear equations

$$AX + XB = C, \quad (5)$$

is called system of fuzzy Sylvester equations if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times m$ crisp matrices and $C = (c_{ij})$ is an $m \times m$ fuzzy matrix.

Definition 6. A fuzzy matrix $X(\alpha) = (x_{ij}(\alpha)) = (\underline{x}_{ij}(\alpha), \bar{x}_{ij}(\alpha))$, $1 \leq i, j \leq m$, $0 \leq \alpha \leq 1$, is called the solution of fuzzy Sylvester equations, if

$$\begin{aligned} \overline{\left(\sum_{k=1}^m a_{ik}x_{kj} + \sum_{k=1}^m x_{ik}b_{kj} \right)}(\alpha) &= \overline{\sum_{k=1}^m a_{ik}x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik}b_{kj}(\alpha)} \\ &= \overline{\sum_{k=1}^m a_{ik}\underline{x}_{kj}(\alpha)} + \overline{\sum_{k=1}^m \underline{x}_{ik}b_{kj}(\alpha)} = \underline{c}_{ij}(\alpha), \\ \overline{\left(\sum_{k=1}^m a_{ik}x_{kj} + \sum_{k=1}^m x_{ik}b_{kj} \right)}(\alpha) &= \overline{\sum_{k=1}^m a_{ik}x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik}b_{kj}(\alpha)} \\ &= \overline{\sum_{k=1}^m a_{ik}\bar{x}_{kj}(\alpha)} + \overline{\sum_{k=1}^m \bar{x}_{ik}b_{kj}(\alpha)} = \bar{c}_{ij}(\alpha). \quad (6) \end{aligned}$$

In particular, if $b_{kj} \geq 0$, $1 \leq k \leq m$, $1 \leq j \leq m$ and $a_{ik} \geq 0$, $1 \leq i \leq m$, $1 \leq k \leq m$, we easily get

$$\begin{aligned} \overline{\left(\sum_{k=1}^m a_{ik}x_{kj} + \sum_{k=1}^m x_{ik}b_{kj} \right)}(\alpha) &= \overline{\sum_{k=1}^m a_{ik}x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik}b_{kj}(\alpha)} \\ &= \sum_{k=1}^m a_{ik}\underline{x}_{kj}(\alpha) + \sum_{k=1}^m \underline{x}_{ik}(\alpha)b_{kj} = \underline{c}_{ij}(\alpha), \end{aligned}$$

$$\overline{\left(\sum_{k=1}^m a_{ik}x_{kj} + \sum_{k=1}^m x_{ik}b_{kj}\right)}(\alpha) = \overline{\sum_{k=1}^m a_{ik}x_{kj}(\alpha)} + \overline{\sum_{k=1}^m x_{ik}b_{kj}(\alpha)} \\ = \sum_{k=1}^m a_{ik}\bar{x}_{kj}(\alpha) + \sum_{k=1}^m \bar{x}_{ik}(\alpha)b_{kj} = \bar{c}_{ij}(\alpha). \quad (7)$$

Now we apply the Kronecker product for solving of system of fuzzy Sylvester equations (SFSE), of the form $AX + XB = C$.

The system of fuzzy Sylvester equations $AX + XB = C$ is equivalent to $AXI + IXB = C$, where

$$(A \otimes I^t)x + (I \otimes B^t)x = c, \quad (8)$$

$$x = (x_{11}, \dots, x_{m1}, \dots, x_{1m}, \dots, x_{mm})^t,$$

$$c = (c_{11}, \dots, c_{m1}, \dots, c_{1m}, \dots, c_{mm})^t.$$

Thus, the system of fuzzy Sylvester equations $AX + XB = C$, is extended to the following crisp block form

$$SZ + ZT = Y, \quad (9)$$

where $S = (s_{ij})$ is an $2m^2 \times 2m^2$ matrix which is defined as follows:

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \quad (10)$$

where S_1 contains the positive entries of $(A \otimes I^t)$ and S_2 contains the absolute values of the negative entries of $(A \otimes I^t)$, $(A \otimes I^t) = S_1 - S_2$. The rest of the entries are zero and $T = (t_{ij})$ is an $2m^2 \times 2m^2$ matrix which is defined as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}, \quad (11)$$

where T_1 contains the positive entries of $(I \otimes B^t)$ and T_2 contains the absolute values of the negative entries of $(I \otimes B^t)$, $(I \otimes B^t) = T_1 - T_2$, and the rest of the entries are zero and

$$Z = \begin{pmatrix} (\underline{x}_{11}) \\ \vdots \\ (\underline{x}_{mm}) \\ -(\bar{x}_{11}) \\ \vdots \\ -(\bar{x}_{mm}) \end{pmatrix}, Y = \begin{pmatrix} (\underline{c}_{11}) \\ \vdots \\ (\underline{c}_{mm}) \\ -(\bar{c}_{11}) \\ \vdots \\ -(\bar{c}_{mm}) \end{pmatrix}. \quad (12)$$

4. Numerical Examples

In this section we illustrate the method from Section 3 by solving some numerical examples.

Example 7. Consider the system of fuzzy Sylvester equations $AX + XB = C$ where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, C = \begin{pmatrix} (0, 1, 2) & (-1, 0, 1) \\ (-1, 0, 1) & (0, 1, 2) \end{pmatrix}.$$

For solving this system, first we transform it to $(A \otimes I^t)x + (I \otimes B^t)x = c$, where

$$(A \otimes I^t) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (I \otimes B^t) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

The extended 8×8 matrices are

$$S = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus:

$$\begin{aligned} x_{11} &= (0.3636\alpha - 1, -0.3636\alpha - 0.2727), \\ x_{12} &= (-0.0909\alpha - 0.3636, -0.5454 + 0.0909\alpha), \\ x_{21} &= (0.1818\alpha + 0.5454, -0.1818\alpha + 0.9091), \\ x_{22} &= (0.4545\alpha - 0.3636, -0.4545\alpha + 0.5454). \end{aligned}$$

Example 8. Consider the system of fuzzy Sylvester equations $AX + XB = C$, where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} (0, 1, 2) & (-2, -1, 0) \\ (-1, 0, 1) & (-1, 1, 2) \end{pmatrix}.$$

For solving this system, first we transform it to $(A \otimes I^t)x + (I \otimes B^t)x = c$, where

$$(A \otimes I^t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}, \quad (I \otimes B^t) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

The extended 8×8 matrices are

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Thus:

$$\begin{aligned} x_{11} &= (0.05 + 0.25\alpha, 0.55 - 0.25\alpha), \\ x_{12} &= (-0.35 + 0.25\alpha, 1.65 - 0.25\alpha), \\ x_{21} &= (0.3125\alpha - 0.5125, 0.1875\alpha - 0.3875), \\ x_{22} &= (0.5625\alpha + 0.8375, 1.4625 - 0.0625\alpha). \end{aligned}$$

5. Conclusion

In this paper, we apply a method for finding solution of system of fuzzy Sylvester equations (SFSE), $AX + XB = C$, where A and B are two $m \times m$ crisp matrices, C is an $m \times m$ fuzzy matrix and X is an $m \times m$ unknown matrix. It is extended to crisp block form $SZ + TZ = Y$ and the linear system $SZ + TZ = Y$ can be solved iff the matrix $(S + T)$ is nonsingular.

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