ON LONG TIME ERROR ANALYSIS TECHNIQUE FOR NERVE AXON TYPE EQUATIONS

Champike Attanayake
Department of Mathematics
Miami University
4200 University Blvd, Middletown, OH – 45042, USA

Abstract: In this paper, long time error estimate for the nerve axon type equations is obtained using a non-traditional method. Traditional methods for analyzing exact error propagation depends on the stability of the numerical method. Whereas, in this paper the analysis of the exact error propagation use attractors which only depends on the stability of the dynamical system. The use of the smoothing indicator yields a posteriori estimates on the numerical error instead of a priori estimates.

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1. Introduction

Consider nerve axon type equations of the form

\[ u_t = D\Delta u + f(u), \]

where \( f \) is twice continuously differentiable function and \( D \) is diagonal matrix with only first element equals to 1. A typical solution of the above system can be given by a traveling wave front, \( u(x, t) = \phi(x - vt) \), where \( v \) is the speed of the traveling wave \( \phi \). These solutions move with constant speed without chang-
ing their shape. Wave solutions of above type arise in numerous problems of
physical interest; such as propagation of nerve impulses, propagation of favor-
able genes, shock waves, and propagation of flames (see [1], [8] and references
therein).

We apply the long time error estimation approach introduced by Sun et
al. [9], [10], to above nerve axon type equations. Particularly, when dealing
with long time error, careful analysis of error propagation is required. Traditi-
onally error analyzes of evolution equations are based on the stability of the
numerical scheme. Under typical conditions, in order for the numerical solution
to converge to the real solution, it is necessary and sufficient to have numeri-
cal stability [7]. Determining the stability of numerical schemes used to solve
complicated non-linear equations is typically difficult and tedious.

To overcome these difficulties Sun and Ewing [9], introduced a long time
error estimation method based on the propagation of the exact error and the
actual local error. The propagation error is estimated using the contraction
properties of the solution to the dynamical system and moving attractors, a
concept which was first introduced in [9]. Moving attractor allows us to analyze
the error prorogation using evolution equation instead of the numerical scheme.
In this paper, for the first time we prove the existence of the moving attractor.
An estimate for the actual error is obtained using smoothing properties of the
numerical scheme.

The outline of the paper is as follows. In Section 2, we use stability of
traveling wave solutions of the nerve axon equation (1), to show existence of
the moving attractor. In Section 3, we define the smoothing indicator for the
nerve axon equation. Finally, in Section 4, computational results of a particular
nerve axon equation are presented.

2. Moving Attractor

In this section we study the existence of the moving attractor for the nerve
axon equation. According to Sattinger [8], if a nerve axon equation has initial
data of the form $u(x, 0) = \phi(x) + h u_0(x)$, for sufficiently small $h$, there exist
constants $K, \omega > 0$ such that

$$\|u(y, t) - \phi_\epsilon(y, t)\|_{L^\infty} \leq Ke^{-\omega t}, \quad (2)$$

for $t \geq 0$ and suitably chosen $\epsilon > 0$. Here $\phi_\epsilon = \phi(y + \epsilon)$, $y = x - vt$.

Let $U(y, t) = u(y + vt, t)$ where $u$ is a solution of (1). Then on the moving
frame we have that
\[
\frac{\partial U}{\partial t} - v \frac{\partial U}{\partial y} - D \Delta U = f(U).
\] (3)

On the other hand, linearization of (3) about \( \phi_{\epsilon} \), leads to the equation
\[
\frac{\partial \tilde{U}}{\partial t} = v \frac{\partial \tilde{U}}{\partial y} + \frac{D}{2} \frac{\partial^2 U_0}{\partial y^2} + \frac{\partial f}{\partial \phi_{\epsilon}}(\phi_{\epsilon}(y)).
\] (4)

The following two lemmas are due to Evens [1] and [2], respectively. Let
\[
\rho(t) = \|U(\cdot, t) - \phi_{\epsilon}(t) - \tilde{U}(\cdot, t)\|_{L^\infty}.
\]

**Lemma 2.1.** If \( \|\tilde{U}(\cdot, t)\|_{L^\infty} \) of (4) is bounded by \( M \) for all \( t \geq 0 \), then
\[
\rho(t) \leq \rho(0)e^{Lt} + \frac{M^2Q}{L}(e^{Lt} - 1), \quad t \geq 0,
\]
where \( L \) and \( Q \) are upper bounds for \( \left| \frac{\partial f(U)}{\partial U} \right| \) and \( \left| \frac{\partial^2 f(U)}{\partial U^2} \right| \), respectively.

**Proof.** This is Lemma 1 of [1]. \( \square \)

The above lemma shows the relation between the linearized and nonlinear solutions of the nerve axon equations.

**Lemma 2.2.** Let \( \tilde{U}(t) \) be the solution of the linearized equation (4) with initial condition \( \tilde{U}(0) \). If there are constants \( a_1, a_2 \) and \( \delta \) such that \( |\delta| \leq a_1\|\tilde{U}(0)\|_{L^\infty} \), then
\[
\left\| \tilde{U}(t) - \delta \left( \frac{d\phi_{\epsilon}}{dy} \right) \right\|_{L^\infty} \leq a_1e^{-a_2t}\|\tilde{U}(0)\|_{L^\infty},
\]
for all \( t > 0 \).

**Proof.** Theorem 1 of [2]. \( \square \)

Now consider some interval \([t, t+s]\), for some fixed \( s > 0 \). If \( \tilde{U}(t+s) \) is a solution of (4) at \( t+s \) with initial value \( \tilde{U}(t) = U(t) - \phi_{\epsilon}(t) \). Note that Lemma 2.2 implies
\[
\left\| \tilde{U}(t+s) - \delta \frac{d\phi_{\epsilon}}{dy}(t+s) \right\|_{L^\infty} \leq a_1e^{-a_2s}\|U(t) - \phi_{\epsilon}(t)\|_{L^\infty} \] (5)
with $|\delta| \leq a_1\|U(t) - \phi_\epsilon(t)\|_{L^\infty}$. And Lemma 2.1 implies that

$$\rho(t + s) \leq \frac{M^2Q}{L}(e^{Ls} - 1).$$

**Theorem 2.3.** Let $U(t) = U(y, t)$ be a solution to the equation (3) with initial data $U(y, 0) = \phi(y) + hU_0(y)$, where $h$ is such that $U(y, t)$ converges to $\phi_\epsilon = \phi(y + \epsilon)$ for some $\epsilon > 0$. Here $\phi$ is the traveling wave solution. Then, for fixed $s > 0$ there exists $T_0$ such that

$$\|U(s + t) - \phi_\delta(s + t)\| \leq 3a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|$$

for all $t > T_0$, for some positive constants $a_1$ and $a_2$.

**Proof.** Let $N$ and $R$ be the upper bounds of $\|d\phi_\epsilon/\delta y\|_{L^\infty}$ and $\|d^2\phi_\epsilon/\delta y^2\|_{L^\infty}$ respectively. Constants $Q$, $L$ and $M$ are defined as in the previous Lemma 2.1. Let $\tilde{U}(t+s)$ is a solution of (4) at $t+s$ with initial value $\tilde{U}(t) = U(t) - \phi_\epsilon(t)$. Now for a fixed $s > 0$, choose $T_0$ such that $Ke^{-\omega t} \leq \min\left(\frac{L}{a_1Qe^{Ls}(1+Ne^{a_2s})^2}, \frac{1}{Ra_1e^{a_2s}}\right)$ for all $t > T_0$, where constants $\omega$ and $K$ defined as in (2) by $\|U(t) - \phi_\epsilon(t)\|_{L^\infty} < Ke^{-\omega t}$. From (5) we have that

$$\|\tilde{U}(t+s)\| \leq \left\|\tilde{U}(t+s) - \delta \frac{d\phi_\epsilon}{\delta y}\right\|_{L^\infty} + \left\|\delta \frac{d\phi_\epsilon}{\delta y}\right\|_{L^\infty}$$

$$\leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} + |\delta|N$$

$$\leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} (1 + Ne^{a_2s})$$

$$:= M. \quad (6)$$

Then, from Lemma 2.1 with (2) and (6),

$$\left\|\left(U - \phi_\epsilon - \tilde{U}\right)(t+s)\right\|_{L^\infty} \leq M^2Q \frac{L}{e^{Ls} - 1}$$

$$= a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \left[a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty}(1 + Ne^{a_2s})^2\frac{Q}{L}(e^{Ls} - 1)\right]$$

$$\leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \left[a_1Ke^{-\omega t}(1 + Ne^{a_2s})^2\frac{Q}{L}e^{Ls}\right]$$

$$\leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \text{ for all } t > T_0. \quad (7)$$
Also,

\[ R\delta^2 \leq Ra_1^2\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \]
\[ \leq a_1\|U(t) - \phi_\epsilon(t)\|_{L^\infty} (Ra_1\|U(t) - \phi_\epsilon(t)\|_{L^\infty}) \]
\[ \leq a_1\|U(t) - \phi_\epsilon(t)\|_{L^\infty}Ra_1Ke^{-\omega t} \]
\[ \leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \quad \text{for all } t > T_0. \tag{8} \]

Since \( R \) is an upper bound for \( d^2\phi_\epsilon/dy^2 \) using (5), (7), (8) and by Taylor expansion, we have that

\[ \|U(t + s) - \phi_\delta(t + s)\|_{L^\infty} \]
\[ \leq \|U - \phi_\epsilon - \tilde{U}\|_{L^\infty} + \|\tilde{U} - \delta \frac{d\phi_\epsilon}{dy}\|_{L^\infty} + \|\phi_\delta - \phi_\epsilon - \delta \frac{d\phi_\epsilon}{dy}\|_{L^\infty} \]
\[ \leq a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} + a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} + Ra_1^2\]
\[ \leq 3a_1e^{-a_2s}\|U(t) - \phi_\epsilon(t)\|_{L^\infty} \]

for all \( t > T_0. \]

For the convenience of error propagation analysis, we use the notation \( u(p, t, v) \) to stand for the value of the solution \( u \) at time \( t + p \) with initial time \( t \), initial value \( v \) and time increment \( p \).

We also need an invariant condition, which guarantees that the absorbing set does not decrease as \( t \to \infty \). If \( M \) is a one-parameter family of sets in \( L^\infty \), \( M = \{ \Phi_t \subset L^\infty | t > T \} \), we say that \( M \) is positively invariant under the dynamical system if for any \( u(t) \in \Phi_t \) and \( p > 0 \), \( u(t + p) \in \Phi_{t+p} \). Following is the definition of the moving attractor given in [10].

**Definition 2.4.** A positively invariant one parameter family of sets \( M \) in \( L^\infty \) is called a moving attractor, if there exists real number \( \theta_s \in (0, 1) \) depending on \( s \), and a one parameter family of open sets \( U = \{ U_t \subset L^\infty | t > T \} \), positively invariant under the dynamical system, with \( \Phi_t \subset U_t \) for all \( t > T \), such that for any \( v \in U_t \)

\[ d(u(s, t, v), \Phi_{t+s}) \leq \theta_s d(v, \Phi_t), \]

where \( d(u, \Phi) = \inf_{w \in \Phi} \|u - w\|_{L^\infty} \). \( U \) is called a basin of the moving attractor.

Following corollary proves the existence of moving attractor for nerve axon equation.
Corollary 2.5. Let \( \phi(x - vt) \) and \( u(x, t) \) be wave profile and solution of nerve axon equation respectively. Define \( \Phi_t \) and \( U_t \): \( \Phi_t = \{ \phi(x - vt + c) | c \in \mathbb{R} \} \), \( U_t = \{ u(x, t), \quad |u(x, 0) = \phi(x) + hu_0(x)\} \). Then the family of sets
\[
\mathcal{M} = \{ \Phi_t | t > T_0 \},
\]
is a moving attractor for the nerve axon equation. The family of sets \( \mathcal{U} = \{ U_t | t > T_0 \} \), is the basin of the moving attractor.

Proof. If \( \phi(x - vt + c) \) is a solution of the nerve axon equation, then \( \phi(x - v(t + s) + c) \) is also a solution to nerve axon equation at time \( t + s \). In other words if \( \phi(x - vt + c) \in \Phi_t \), then \( \phi(x - v(t + s) + c) \in \Phi_{t+s} \). Therefore, the family of sets \( \mathcal{M} \) is positively invariant under the nerve axon equation. Clearly the family of sets
\[
\mathcal{U} = \{ U_t | t > T_0 \},
\]
is also positively invariant under the nerve axon equation and \( \Phi_t \subset U_t \). Then, under appropriately chosen \( s \), that is \( s > \frac{1}{a_2} \ln 3a_1 \) when \( a_1 > 1/3 \), \( 3a_1 e^{-a_2 s} = \theta_s \in (0, 1) \), such that
\[
\| u(s, t, v) - \phi_{\gamma}(t + s) \|_{L^\infty} \leq \theta_s \| v - \phi_{\epsilon}(t) \|_{L^\infty},
\]
for any \( v \in U_t \). Thus
\[
d(u(s, t, v), \Phi_{t+s}) \leq \theta_s d(v, \Phi_t). \quad \square
\]

3. Smoothing Indicator

In this section we define smoothing indicator for the nerve axon equation. Numerical smoothing is a property of the computed numerical solution. To solve nerve axon equation numerically we use finite element method. We discretize the problem, first in the spatial variable \( x \), which results in an approximate solution \( u_h \) in the finite element space \( V_h \), as a solution of an initial value problem for a finite-dimensional system of ordinary differential equations (ODE). We then use a finite difference time stepping method to solve this finite dimensional ODE [11].

We recall that semigroup operator \( E(t) = e^{D\Delta t} \) is the solution operator of the initial value problem for the homogenous equation
\[
u' - D\Delta u = 0, \quad u(t) = \bar{u},
\]
defined in the interval \([t, t + s]\). Its solution is thus given by \(u(t + s) = E(s)u(t)\). By Duhamel’s principle it follows that for the corresponding semi-linear equation

\[
u' - \mathcal{D} \Delta u = f(u), \quad u(t) = \bar{u},
\]

the solution may be written as

\[
u(s, t, \bar{u}) = E(s)\bar{u} + \int_t^{t+s} E(t + s - r) f(u(r, t, \bar{u})) dr.
\] (9)

For the proofs in this section we introduce the discrete Laplacian \(\Delta_h : H^1_0 \to V_h\) defined by \((\Delta_h u, v) = -(\nabla u, \nabla v)\) and \(L^2\) projection operator \(P_h : L^2 \to V_h\) defined by \((P_h u, v) = (u, v)\) for all \(v \in V_h\). Then the discrete version of the nerve axon equation becomes

\[
u_h' = \mathcal{D} \Delta_h u_h + P_h f(u_h)
\] (10)

with initial value \(\bar{u}\). The solution can be written using the semigroup operator \(E_h(t) = e^{\mathcal{D} \Delta_h t}\) as

\[
u_h(s, t, \bar{u}) = E_h(s)\bar{u} + \int_t^{t+s} E_h(t + s - r) P_h f(u_h(r, t, \bar{u})) dr.
\] (11)

Then from [11] and using the fact that, solution \(u_h\) tend to 0 in the region there are no pulse, we have that

\[
\|E_h(s)\nu\|_{L^\infty} \leq C_\omega l_h \|\nu\|_{L^\infty},
\] (12)

where \(C_\omega\) is constant and \(l_h = \max(1, \log 1/h)\) with mesh size \(h\).

The following theorem establishes a smoothing property for the estimation of the local error resulting from the discretization of time.

**Theorem 3.1.** For any initial value \(\bar{u} \in V_h\) at time node \(t_n\), if

\[
\begin{align*}
\tilde{v} &= \mathcal{D} \Delta_h \bar{u} + P_h f(\bar{u}) \\
\bar{w} &= \mathcal{D} \Delta_h \bar{v} + P_h f'(\bar{u}) \bar{v}
\end{align*}
\]

and \(\|\bar{u}\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\bar{w}\|_{L^\infty}\) are bounded, then the semi-discrete solution \(u_h(s, t_n, \bar{u})\) satisfies

\[
\|u_h''\|_{L^\infty} \leq N_{s, \bar{v}, \bar{w}}^s,
\]

where \(N_{s, \bar{v}, \bar{w}}^s\) is a computable function depends on \(\bar{u}, \tilde{v}, \bar{w}\) and \(s\).
Proof. Let \( v_h = du_h/dt = u'_h \) and \( w_h = d^2u_h/dt^2 = v'_h = u''_h \). That is,

\[
\begin{align*}
v'_h & = D\Delta_h v_h + P_h f'(u_h)v_h, \\
w'_h & = D\Delta_h w_h + P_h f'(u_h)w_h + P_h f''(u_h)v^2_h.
\end{align*}
\]

The solutions of the above two equations satisfy

\[
v_h(s, t, \bar{v}) = E_h(s)\bar{v} + \int_t^{t+s} E_h(t + s - r)P_h f'(u_h(r, t, \bar{u}))v_h(r, t, \bar{v})dr, \\
w_h(s, t, \bar{w}) = E_h(s)\bar{w} + \int_t^{t+s} E_h(t + s - r)(P_h f''(u_h(r, t, \bar{u}))v^2_h(r, t, \bar{v}) + P_h f'(u_h(r, t, \bar{u})))w_h(r, t, \bar{w})dr.
\]

Also, note that

\[
\| f^{(q)}(u_h(r)) \|_{L^\infty} \leq \| f^{(q)}(\bar{u}) \|_{L^\infty} + \| f^{(q)}(u_h(r, t, \bar{u})) - f^{(q)}(\bar{u}) \|_{L^\infty} \\
\leq \| f^{(q)}(\bar{u}) \|_{L^\infty} + L\| u_h(r, t, \bar{u}) - \bar{u} \|_{L^\infty},
\]

for some constant \( L \), where \( q = 0, 1, 2 \). Then for any \( 0 < p < s \), from (11) and Gronwall’s lemma, we have that

\[
\begin{align*}
\| u_h(p, t, \bar{u}) \|_{L^\infty} & \leq C_{\omega} l_h \| \bar{u} \|_{L^\infty} + C_{\omega} l_h \int_t^{t+p} \| f(u_h(r, t, \bar{u})) \|_{L^\infty} dr \\
& \leq C_{\omega} l_h \| \bar{u} \|_{L^\infty} + C_{\omega} l_h \int_t^{t+p} (\| f(\bar{u}) \|_{L^\infty} + L\| \bar{u} \|_{L^\infty}) \\
& \quad + C_{\omega} l_h L \int_t^{t+p} \| u_h(r, t, \bar{u}) \|_{L^\infty} dr \\
& \leq C_{\omega} l_h (1 + Lp)\| \bar{u} \|_{L^\infty} + C_{\omega} l_h p\| f(\bar{u}) \|_{L^\infty} \int_t^{t+p} \| u_h(r, t, \bar{u}) \|_{L^\infty} dr \\
& \leq (C_{\omega} l_h (1 + Lp))\| \bar{u} \|_{L^\infty} + C_{\omega} l_h p\| f(\bar{u}) \|_{L^\infty} \exp(pC_{\omega} l_h L) \\
& \quad + (C_{\omega} l_h (1 + Ls))\| \bar{u} \|_{L^\infty} + C_{\omega} l_h s\| f(\bar{u}) \|_{L^\infty} \exp(sC_{\omega} l_h L) \\
& = N^s_{\bar{u}}
\end{align*}
\]
Using the same argument now on (13) for $0 < p \leq s$,
\[
\|v_h(p, t, \bar{v})\|_{L^\infty} \leq C_\omega l_h \|\bar{v}\|_{L^\infty} + C_\omega l_h \int_t^{t+p} \|f'(u_h(r, t, \bar{u}))v_h(r, t, \bar{v})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{v}\|_{L^\infty} + C_\omega l_h \int_t^{t+p} (\|f'(\bar{u}) + L(u_h(r, t, \bar{u}) - \bar{u})\|_{L^\infty} v_h(r, t, \bar{v})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{v}\|_{L^\infty} + C_\omega l_h \int_t^{t+p} (\|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|_{L^\infty} v_h(r, t, \bar{v})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{v}\|_{L^\infty} \exp(C_\omega l_h \int_t^{t+p} \|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{v}\|_{L^\infty} \exp(C_\omega l_h p \|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|_{L^\infty})
\]
\[
\leq C_\omega l_h \|\bar{v}\|_{L^\infty} \exp(C_\omega l_h s \|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|_{L^\infty})
\]
\[
= N^s_{\bar{u}, \bar{v}}.
\]

(15)

Then by (14)
\[
\|w_h(s, t, \bar{w})\|_{L^\infty} \leq C_\omega l_h \|\bar{w}\|_{L^\infty}
\]
\[
+ C_\omega l_h \int_t^{t+s} \|(f''(u_h(r, t, \bar{u}))v_h^2(r, t, \bar{v}) + f'(u_h(r, t, \bar{u}))w_h(r, t, \bar{w})\|_{L^\infty} \, dr
\]
\[
\leq Cl_h \|\bar{w}\|_{L^\infty} + C_\omega l_h \int_t^{t+s} (\|f''(\bar{u}) + L(u_h(r, t, \bar{u}) - \bar{u})\|v_h^2(r, t, \bar{v})\|_{L^\infty} \, dr
\]
\[
+ C_\omega l_h \int_t^{t+s} (\|f'(\bar{u}) + L(u_h(r, t, \bar{u}) - \bar{u})\|w_h(r, t, \bar{w})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{w}\|_{L^\infty} + C_\omega l_h \int_t^{t+s} (\|f''(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\)(N^s_{\bar{u}, \bar{v}})^2\|_{L^\infty} \, dr
\]
\[
+ C_\omega l_h \int_t^{t+s} (\|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|w_h(r, t, \bar{w})\|_{L^\infty} \, dr
\]
\[
\leq C_\omega l_h \|\bar{w}\|_{L^\infty} + C_\omega l_h (\|f''(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\)(N^s_{\bar{u}, \bar{v}})^2\|_{L^\infty}
\]
\[
+ \exp(C_\omega s l_h (\|f'(\bar{u}) + L(N^s_{\bar{u}} - \bar{u})\|_{L^\infty})
\]
\[
= N^s_{\bar{u}, \bar{v}, \bar{w}}.
\]

**Definition 3.2.** The smoothing indicator at the time node $t_n$ is defined as $S_n = (N^s_{\bar{u}, \bar{v}, \bar{w}}, \Delta_h \bar{u})$, where $\bar{u} = u_N(t_n), \bar{v} = D \Delta_h \bar{u} + P_h \bar{f}(\bar{u})$ and $\bar{w} = D \Delta_h \bar{v} + P_h (\nabla f'(\bar{u}) \bar{v})$. Here $u_N$ is fully desecrate numerical solution.

If both components of $S_n$ are bounded at each node $t_n$, we say that the numerical solution is smooth and stable. The term $N^s_{\bar{u}, \bar{v}, \bar{w}}$ of the smoothing
indicator will be used to monitor the smoothness of the numerical solution as a function of time $t$, while the term $\Delta_h u_n$ is used to monitor the smoothness of the numerical solution as a function of spacial variable $x$.

Now we can estimate global error of the nerve axon equation (1) from the properties of the moving attractor and the stability smoothing indicator using the theorem 6.1 developed by Sun and Fillipova in [10].

**Theorem 3.3.** Let $u$ be the exact solution of the nerve axon equation (1) and $u_N$ be the numerical solution. Assume that: (a) the stability smoothing indicator as defined in definition (3.2) remains bounded, (b) the time step $\tau$ is chosen so that $s$ is a multiple of $\tau$: $s = k\tau$, $k = 1, 2, \ldots$. Then for any node from $T_0$ to $\infty$ of the form $T_0 + ns$, we have the following global error estimate:

$$d(\Phi_{t_0+ns}, u_N(ns, T_0, u_N(T_0))) \leq C h^2 l_h^2 S_H + \tau s S_M + \theta_s^n d(\Phi_{T_0}, u_N(T_0)).$$

Here $S_M = \max_n S_{n,1}$, and $S_H = \max_n S_{n,2}$, where $S_{n,1}$ and $S_{n,2}$ denote the first and second components of smoothing indicator $S_n$ respectively.

**Proof.** Theorem 6.1, [10].

4. Numerical Experiment

Consider FitzHugh’s nerve axon equation [5],

$$\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + 10(u_1 - \frac{1}{3}u_1^3 - u_2) \\
\frac{\partial u_2}{\partial t} &= 0.8(1.5 + 1.25u_1 - u_2) \\
u_1(0) &= -1.5 \quad u_2(0) = -\frac{3}{8}.
\end{align*}$$

To obtain the numerical solution we used piecewise linear elements and backward Euler method. Figure 1 shows the traveling wave front of a nerve impulse at time $t = 6, 7, 8, 9, \ldots, 13, 14$. Consistency of shape of the wave front indicates that the error of the numerical solution is not growing with time. Figure 2 shows the first component of the smoothing indicator. Throughout the computation smoothing indicator remain bounded. It is clear from Figure 1 that the second component of the stability smoothing indicator remains bounded.
Figure 1: Traveling wave fronts of FitzHugh’s nerve axon equation.

References


Figure 2: First component of smoothing indicator for FitzHugh’s nerve axon equation.

