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OPTIMAL NET INVESTMENT WEALTH WITH DISCOUNTED STOCHASTIC CASH FLOWS AND EFFICIENT FRONTIER

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Abstract: This paper study the optimal net investment wealth with discounted stochastic cash flows for a certain investor who trades in complete diffusion models, receives a stochastic cash inflows and pays a stochastic cash outflows. The cash inflows and cash outflows are correlated to the index bond and stock market risks over time. This paper aims at determining the optimal variational Merton portfolio, expected final net investment wealth and its variance and efficient frontier for the three classes of assets: stock, index bond and cash account. The discounted cash inflows and outflows are obtained. The dynamics of the net investment wealth process involving two risky assets and a cash account is obtained. We further established the market efficiency test and efficient frontier for the final net investment wealth process of the investor. We obtain the optimal terminal net investment wealth and show that the discounted cash inflows and cash outflows depend on the optimal wealth of the investor. The expected terminal net investment wealth with zero variance is established. Some numerical results were illustrated in this paper.

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1. Introduction

In this paper, we study the expected final net investment wealth and the variance of the final net investment wealth for an investor. Again, the discounted cash inflows and cash outflows were found to have been influented by financial market behavior. This, in one hand, help investors to know how to invest their funds, since market behavior can encourage (or discourage) investors to invest their resource. On the other hand, the cash outflows will increase when the financial market reacts positively (booming) and decrease when the market reacts negatively (bearish). This cash inflows are expected to be gradually transfer to cash account over time. Note that when the financial market reacts positively, the outflows (either as a part or a whole) may be re-invested into the markets. In that case, we say that the such fund is borrowed to finance the risky assets.

This paper considers the optimal portfolios and net investment wealth of an investor, who receives continuous-time stochastic cash inflows and pays continuously a stochastic cash outflows. The cash inflows are invested into a cash account, an index bond and a stock. A compact form of the wealth process of the investor was established in this paper. This paper intends to determine the optimal variational Merton portfolios [22], [23] (in stock and index bond) and the terminal net investment wealth that will accrued to the investor. The discounted cash inflows was found to depend only on the risks associated with the financial market while the discounted cash outflows depend on its own risks and the risk associated with the financial market as well.

In related literature, Davis [11] examined the rational, nature and financial consequences of two alternative approaches to portfolio regulations for the long-term institutional investor sectors of life insurance and pension funds. Brawne *et al* [5] developed a model for analyzing the *ex ante* liquidity premium demanded by the holder of an "illiquid annuity". The annuity is an insurance product that is similar to a pension savings account with both an accumulation and "decumulation" phase. They computed the yield needed to compensate for the utility welfare loss, which is induced by the inability to re-balance and maintain an optimal portfolio when holding an annuity. Deelstra *et al* [13], [14], [15] considered the optimal design of the minimum guarantee in a defined contribution pension fund scheme. They studied the investment in the financial market by assuring that the pension fund optimizes its retribution which is a part of the surplus, that is the difference between the pension fund value and the guarantee. Jensen and Sørensen [18] measured the effect of a minimum interest rate guarantee constraint through the wealth equivalent in case of no constraints and show numerically the guarantees may induce a significant utility loss for relative risk tolerant investors. Cairns *et al* [7] developed a pension plan accumulation programmed designed to deliver a pension in retirement that is closely related to salary that the plan member received prior to retirement. Cairns *et al* [7] considered the finding of the optimal dynamic asset allocation strategy for a defined contribution (DC) pension plan, taking into account the stochastic features of the plan members lifetime salary progression as well as the stochastic properties of the assets held in his accumulating pension fund. They emphasized that salary risk (the fluctuation in the plan members earning in response to economic shocks) is not fully hedgeable using existing financial assets. They further emphasized that wage-indexed bonds could be used to hedge productivity and inflation shocks, but such bonds are not widely traded. They called the optimal dynamic asset allocation strategy stochastic life-styling. They compare it against various static and deterministic lifestyle strategies in order to calculate the costs of adopting suboptimal strategies. According to Cairns *et al* [8], the solution technique of Cairns *et al* [7], made using of the present value of future contribution premiums into the plan. This technique can also be found in Boulier *et al* [4], Deelstra *et al* [12], Korn and Krekel [19] and Blake *et al* [3]. Deterministic life-styling which is the gradual switch from equities to bonds according to present rules during the accumulation phase of DC pension plans and was designed to protect the pension fund from a catastrophic fall in the stock market just prior to retirement, see Cairns *et al* [7], Blake *et al* [3] and Cajueiro and Yoneyama [9]. Haberman and Vigna [16] and Cairns *et al* [6], [7] analyzed extensively the occupational DC pension fund, where the contribution rate is a fixed percentage of salary. Battocchio and Menoncin [1] used a stochastic dynamic programming approach to model a DC pension fund in a complete financial market with stochastic investment opportunities and two background risks: salary risk and inflation risk. They gave a closed form solution to the asset allocation problem and analyze the behavior of the optimal portfolio with respect to salary and inflation. Many authors considered the dynamics of cash inflows of an investor in different forms. For example, for a constant flow of contributions (or cash inflows), see Højgaard and Vigna [17]. For stochastic cash inflows, see Maurer *et al* [21], Battocchio [2], Zhang *et al* [28], Zhang [27], Korn and Kruse [20]. Maurer *et al* [21] modeled inflation index that involves inflation uncertainty. They considered multi-decade investment horizons. Zhang *et al* [28] considered the optimal management and inflation protection strategy for defined contribution pension plans using Martingale approach. They derived an analytical expression for the optimal strategy and expresses it in terms of observable market variables. Dai *et al* [10] studied a continuous-time Markowitz's mean-variance portfolio selection problem involving propositional transaction costs. They established a critical length of time which depends on the stock excess return as well as the transaction fees but independent of the investment target and stock volatility. Nkeki [24], and Nkeki and Nwozo [25] studied the variational form of classical portfolio strategy and expected wealth for a pension plan member. They assumed that the growth rate of salary is a linear function of time and that the cash inflow is stochastic. Nkeki and Nwozo [26], studied the optimal portfolio strategies with stochastic cash flows and expected optimal terminal wealth under inflation protection for a certain Investment Company who trades in a complete diffusion model, receives a stochastic cash inflows and pays a stochastic outflows to its holder. They found that as the market evolve parts of the index bond and stock portfolio values should be transferred to cash account. This, to a great extent will protect the IC from catastrophic fall in the stock market. They also found that the portfolio processes involved inter-temporal hedging terms that offset any shock to both the stochastic cash inflows and cash outflows.

The issue of investing stochastic cash inflows in various investment settings (such as pension scheme) has been studied in most of the mentioned literature. But, the issue of investing stochastic cash inflows and received a stochastic cash outflows (such as pension scheme with flows of minimum pension benefits) and efficient frontier of the resulting net investment wealth have not been studied in the literature of financial mathematics and insurance, to the best of our knowledge. Hence, it is one of the motivations of this study. The other aims of this paper are to determine: net investment wealth and expected net investment wealth of an investor, the market efficient test and the efficient frontier for the three classes of assets: stock, index bond and cash account.

Remark 1. In this paper, the net investment wealth, discounted cash inflows and discounted cash outflows could be seen, in the contest of pension scheme, respectively as surplus, discounted contributions and flow of minimum pension benefits. In particular, this may be a special case of pension plan with minimum pension benefits. Though, the results in this paper are also more general.

The remainder of this paper is organized as follows. In Section 2, we present the financial market models. The wealth process of the investor is presented in Section 3. Section 4 presents the expected discounted cash inflows and cash outflows of the investor. The net investment wealth of the investor is presented in Section 5. In Section 6, we present the Hamilton-Jacobi-Bellman equation and optimal portfolio strategies for the investor. In Section 7, we present the optimal net investment wealth and discounted cash flows that dependent on optimal wealth of the investor. Finally, Section 8 concludes the paper.

2. The Financial Model

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0,T]\}$, where $\mathcal{F}_t = \sigma(W^S(s), W^I(s) : s \leq t)$, where $W^S(t)$ and $W^I(t)$ are Brownian motions with respect to stock and index bond at time t. Let $S(t)$ be the stock price process at time $s \leq t$ and $I(t)$ the price index at time $s \leq t$. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ are 2-dimensional processes, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure and $\sigma^S(t) = (\sigma_1^S(t), \sigma_2^S(t))$ and $\sigma_I(t) = (\sigma_1(t), 0)$ are the volatility of stock and volatility of the index bond with respect to changes in $W^S(t)$ and $W^I(t)$, respectively, where $\sigma_1^S(t)$ is volatility of the stock arising from index bond risk, $\sigma_2^S(t)$ is volatility of the stock arising from the stock market risk, $\sigma_1(t)$ is volatility of the index bond. $\mu(t)$ is the appreciation rate for stock. Moreover, $\sigma^S(t)$ and $\sigma_I(t)$ are the volatilities for the stock and index bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

We assume that the investor faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are tradeable. In this paper, we allow the stock price to be correlated to index bond risk. Also, we correlated the cash inflows and outflows to stock and index bond market risks in other to determine the extent to which cash inflows and outflows should be hedged. The dynamics of the underlying assets are given by (1) to (3):

$$dB(t) = r(t)B(t)dt, B(0) = 1; \tag{1}$$

$$dS(t) = \mu(t)S(t)dt + \sigma_1^S(t)S(t)dW^I(t) + \sigma_2^S(t)S(t)dW^S(t), S(0) = s_0 > 0; \tag{2}$$

$$dF(t, I(t)) = (r(t) + \sigma_1(t)\theta^I(t))F(t, I(t))dt + \sigma_I(t)F(t, I(t))dW(t), \quad F(0) = F_0 > 0; \tag{3}$$

where, $r(t) \in \mathcal{R}_+$ is the nominal interest rate, $\theta^I(t) \in \mathcal{R}$ is the price of index bond risk, $B(t)$ is the price process of the cash account at time t, $S(t)$ is stock price process at time t, $I(t)$ is the price index at time t and has the dynamics: $dI(t) = q(t)I(t)dt + \sigma_1(t)I(t)dW^I(t)$, where $q(t) \in \mathcal{R}$ the expected price index, which is the difference between nominal interest rate. $F(t, I(t))$ is a zero-coupon bond which pays the price index at maturity, with a payoff

$$F(t, I(t)) = E_t\left[I(T)\frac{\Lambda(T)}{\Lambda(t)}\right],$$

where

$$\Lambda(t) = B(t)^{-1}Z(t)$$

and $Z(t)$ satisfies the process

$$Z(t) = \exp\left(-\theta'(t)W(t) - \frac{1}{2}\|\theta(t)\|^2 t\right), \tag{4}$$

which we assume to be Martingale in $\mathbf{P}$.

Then, the volatility matrix

$$\Sigma(t) := \begin{pmatrix} \sigma_1(t) & 0 \\ \sigma_1^S(t) & \sigma_2^S(t) \end{pmatrix} \tag{5}$$

corresponds to the two risky assets, $S(t)$ and $F(t, I(t))$, and it satisfies $det(\Sigma(t)) = \sigma_1(t)\sigma_2^S(t) \neq 0$. Therefore, the market is complete and there exists a unique market price $\theta(t)$ (the vector of standardized risk premia or Sharpe ratio of the investor's portfolio) satisfying

$$\theta(t) := \begin{pmatrix} \theta^I(t) \\ \theta^S(t) \end{pmatrix} = \begin{pmatrix} \theta^I(t) \\ \dfrac{\mu(t) - r(t) - \theta^I(t)\sigma_1^S(t)}{\sigma_2^S(t)} \end{pmatrix}, \tag{6}$$

where $\theta^S(t)$ is the market price of stock risks. We assume in this paper that the cash inflows process $\varphi(t)$ at time t and cash outflows process $\Gamma(t)$ at time t follow the dynamics, respectively presented in (7) and (8):

$$d\varphi(t) = \varphi(t)(\omega dt + \sigma_\varphi dW(t)), \varphi(0) = \varphi_0 \in \mathcal{R}_+ \tag{7}$$

$$d\Gamma(t) = \Gamma(t)(\delta dt + \sigma_\Gamma dW(t)), \Gamma(0) = \Gamma_0 \in \mathcal{R}_+ \tag{8}$$

where, $\sigma_\varphi = (\sigma_1^\varphi, \sigma_2^\varphi)$, $\sigma_\Gamma = (\sigma_1^\Gamma, \sigma_2^\Gamma)$, $\omega \in \mathcal{R}_+$ is the expected growth rate of the cash inflows and $\sigma_1^\varphi \in \mathcal{R}$ is the volatility of the cash inflows caused by the source of risk, $W^I(t)$ and $\sigma_2^\varphi \in \mathcal{R}$ is the volatility cash inflow caused by the source of uncertainty arises from the stock market, $W^S(t)$, and $\delta \in \mathcal{R}_+$ is the expected growth rate of the cash outflows and $\sigma_1^\Gamma \in \mathcal{R}$ is the volatility cash outflows caused by the source of risk, $W^I(t)$ and σ_2^Γ is the volatility cash outflows caused by the source of uncertainty arises from the stock market, $W^S(t)$.

Applying Itô Lemma on the stochastic differential equations (7) and (8), we have the following:

$$\varphi(t) = \varphi_0 \exp((\omega - \frac{1}{2}\|\sigma_\varphi\|^2)t + \sigma_\varphi W(t)), \tag{9}$$

$$\Gamma(t) = \Gamma_0 \exp((\delta - \frac{1}{2}\|\sigma_\Gamma\|^2)t + \sigma_\Gamma W(t)). \tag{10}$$

3. The Wealth Process of the Investor

Let $X^{\varphi\Gamma}(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the index bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t. We assume in this paper that $r(t)$, $\mu(t)$, $\sigma_1(t)$, $\sigma_2^S(t)$, $\sigma_1^S(t)$, $\theta^I(t)$, $\theta^S(t)$ are constants in time. Therefore, the fund dynamic evolution under the investment policy Δ is given by

$$\begin{aligned} &dX^{\varphi\Gamma}(t) = X^{\varphi\Gamma}(t)[(r + \Delta(t)A)dt + (\Sigma'\Delta'(t))'dW(t)] + (\varphi(t) - \Gamma(t))dt, \\ &X^{\varphi\Gamma}(0) = x_0^\Gamma \in \mathcal{R}_+, \end{aligned} \tag{11}$$

where $'$ denotes transpose, $A = (\sigma_1\theta^I, \mu - r)'$ and $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix}$. If the wealth process does not involve cash outflows, then, (11) becomes

$$\begin{aligned} &dX^{\varphi}(t) = X^{\varphi}(t)[(r + \Delta(t)A)dt + (\Sigma'\Delta'(t))'dW(t)] + \varphi(t)dt, \\ &X^{\varphi}(0) = x_0^\varphi \in \mathcal{R}_+. \end{aligned} \tag{12}$$

If the wealth process does not involve both cash inflows and cash outflows, then, (11) becomes

$$\begin{aligned} &dX(t) = X(t)[(r + \Delta(t)A)dt + (\Sigma'\Delta'(t))'dW(t)], \\ &X(0) = x_0 \in \mathcal{R}_+. \end{aligned} \tag{13}$$

Observe that, we can express (11) as

$$dX^{\varphi\Gamma}(t) = X^{\varphi\Gamma}(t)\frac{dX(t)}{X(t)} + (\varphi(t) - \Gamma(t))dt$$

and

$$dX(t) = dX^{\varphi\Gamma}(t) - (\varphi(t) - \Gamma(t))dt.$$

4. The Expected Value of Discounted Stochastic Cash Flows

In this section, we consider the valuation of discounted future cash inflows and cash outflows at time t.

4.1. The Expected Value of Discounted Stochastic Cash Inflows

Definition 1. The expected discounted future stochastic cash inflows process is defined as

$$\Psi(t) = E\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\varphi(u)du|\mathcal{F}(t)\right], \tag{14}$$

where, $\Lambda(t) = \dfrac{Z(t)}{B(t)} = \exp(-rt)Z(t)$ is the stochastic discount factor which adjusts for nominal interest rate and market price of risks, and $E(.|\mathcal{F}(t))$ is a real world conditional expectation with respect to the Brownian filtration $(\mathcal{F}(t))_{t\geq 0}$. For details on real world measure $\mathbf{P}$, see Zhang [27], Nkeki [24], Nkeki and Nwozo [25].

Theorem 1. *Let $\Psi(t)$ be the expected discounted future stochastic cash inflows (EVDFSCI) process, then*

$$\Psi(t) = \frac{\varphi(t)}{\phi}(\exp[\phi(T-t)] - 1), \tag{15}$$

where $\phi = \omega - r - \sigma_\varphi\theta$.

Proof: See Nkeki [24].

Theorem 1 tells us that the expected discounted future stochastic cash inflows process $\Psi(t)$ is proportional to the instantaneous total stochastic cash inflows process $\varphi(t)$. Observe that at time T, the value of the inflow of cash is zero. This is because the value Ψ_0 has been invested while setting up the investment.

Taking a differential of both sides of (15), we obtain

$$d\Psi(t) = \Psi(t)[(r + \sigma_1^{\varphi}\theta^I + \sigma_2^{\varphi}\theta^S)dt + \sigma_1^{\varphi} dW^I(t) + \sigma_2^{\varphi} dW^S(t)] - \varphi(t)dt. \quad (16)$$

4.2. The Expected Value of Discounted Stochastic Cash Outflows

Definition 2. The expected discounted flow of stochastic cash outflows process at time t is defined as

$$\Phi(t) = E\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)}\Gamma(u)du|\mathcal{F}(t)\right], T \geq t. \quad (17)$$

The contingent claim $\Gamma(t)$ that matures at the stopping time $t \in [0, T]$ is an $\mathcal{F}(t)$-measurable non-negative payoff that possesses a finite expectation. As outlined in [7], the value $\Phi(t)$ can be obtained at time t by the real-world pricing formula given in (17).

Theorem 2. *Let $\Phi(t)$ be the expected discounted cash outflows (EDCO) process, then*

$$\Phi(t) = \frac{\Gamma(t)}{\beta}\left(1 - \exp[-\beta T]\right), \quad (18)$$

where $\beta = \delta - r - \sigma^{\Gamma}\theta$.

Proof: See Nkeki [24].

Theorem 2 tells us that the expected discounted stochastic cash outflows process $\Phi(t)$ is proportional to the instantaneous total stochastic cash outflows process $\Gamma(t)$.

Taking a differential of both sides of (18), we have

$$d\Phi(t) = \Phi(t)(\alpha dt + {\sigma_1}^{\Gamma} dW^I(t) + {\sigma_2}^{\Gamma} dW^S(t)) - \Gamma(t)dt, \quad (19)$$

where, $\alpha = \dfrac{\delta(1 - \exp(-\beta T)) + \beta}{1 - \exp(-\beta T)}$.

5. The Net Investment Wealth for the Investor

This section considers the investor's net investment wealth at time t as well as its existence and uniqueness.

Definition 3. The net investment wealth process of the investor at time t is defined as

$$V(t) = X^{\varphi\Gamma}(t) + \Psi(t) - \Phi(t). \tag{20}$$

The net investment wealth $V(t)$ equals the wealth process $X^{\varphi\Gamma}(t)$ plus the expected discounted future stochastic cash inflows, $\Psi(t)$, less the expected discounted stochastic cash outflows $\Phi(t)$.

Proposition 1. *Suppose that Definition 3 holds and $X^{\varphi,\Gamma}(t)$, $\Psi(t)$ and $\Phi(t)$ satisfy (11), (16) and (19), respectively, then the dynamics of the net investment wealth of the investor is*

$$dV(t) = (rV(t) + \bar{y}D(t))dt + \bar{y}H(t)dW(t)], V(0) = v_0, \tag{21}$$

where $D(t) = (\Delta(t)A, \sigma_\varphi\theta, -r-\alpha)'$; $H(t) = ((\Sigma'\Delta'(t))', \sigma_\varphi, -\sigma_\Gamma)'$, $\bar{y} = (X(t), \Psi(t), \Phi(t))$.

Proof. We find the differential of both sides of (20), substitute in (11), (16) and (19), and carry out some transformation, then the result follows. The reason for introducing the notations $D(t)$, $H(t)$ and $\bar{y}$ is to enables us to write the differential of (20) in compact form. □

6. Optimal Portfolio as a Function of Cash Inflows and Cash Outflows

In this section, we consider the optimal portfolio process for the investor which is a function cash flows processes. We define the general objective function

$$J_1(t, v) = E[U(V(T))|V(t) = v],$$

where $J_1(t, v)$ is the path of $V(t)$ given the portfolio strategy $\Delta(t) = (\Delta^I(t), \Delta^S(t))$. Define $A(v)$ to be the set of all admissible portfolio strategies that are $\mathcal{F}_v$-progressively measurable, satisfying

$$E\int_0^T \Delta(t)\Delta'(t)dt < \infty,$$

and let $U(t, v)$ be a concave value function in $V(t)$ such that

$$U(t, v) = \sup_{\Delta\in A(v)} J_1(t, v). \tag{22}$$

Then $U(t,v)$ satisfies the HJB equation

$$\begin{aligned} &U_t + \sup_{\Delta\in A(v)} \mathcal{HV}(t,v) = 0, \\ &\text{subject to: } U(T,v) = \frac{1}{\gamma}v^{\gamma}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \mathcal{HV}(t,v) &= (rx + \Delta(t)Ax)U_x + (r\Psi + \Psi\sigma_{\varphi}\theta)U_{\Psi} - \alpha\Phi U_{\Phi} \\ &+\frac{1}{2}(\Sigma'\Delta'(t))'\Sigma'\Delta'(t)x^2U_{xx} + x\Psi(\Sigma'\Delta'(t))'\sigma'_{\varphi}U_{x\Psi} - x\Phi(\Sigma'\Delta'(t))'\sigma'_{\Gamma}U_{x\Phi} \\ &+\frac{1}{2}\Psi^2\sigma_{\varphi}\sigma'_{\varphi}U_{\Psi\Psi} - \Psi\Phi\sigma_{\varphi}\sigma'_{\Gamma}U_{\Psi\Phi} + \frac{1}{2}\Phi^2\sigma_{\Gamma}\sigma'_{\Gamma}U_{\Phi\Phi}. \end{aligned} \tag{24}$$

Since U is a concave function in $V(t)$ and $U(t,v) \in C^{1,2}(R \times [0,T])$, then (23) is well defined. Finding the partial derivative of $\mathcal{HV}(t,v)$ with respect to $\Delta(t)$ and set to zero, we have

$$(\Delta'(t))^* = -\frac{(\Sigma\Sigma')^{-1}A}{x}\frac{U_x}{U_{xx}} - \frac{\Psi\Sigma^{-1}\sigma'_{\varphi}U_{x\Psi}}{xU_{xx}} + \frac{\Phi\Sigma^{-1}\sigma'_{\Gamma}U_{x\Phi}}{xU_{xx}}. \tag{25}$$

This is the variational form of Merton portfolio value. Substituting (25) into (23), we have

$$\begin{aligned} &U_t + rxU_x + r\Psi U_{\Psi} - \alpha\Phi U_{\Phi} + \Psi\sigma_{\varphi}\theta U_{\Psi} - \frac{(M_A)'AU_x^2}{U_{xx}} - \\ &\frac{\Psi(M_{\varphi})'AU_xU_{x\Psi}}{U_{xx}} + \frac{\Phi(M_{\Gamma})'AU_xU_{x\Phi}}{U_{xx}} + \frac{(\Sigma'M_A)'(\Sigma'M_A)U_x^2}{2U_{xx}} \\ &+\frac{\Psi(\Sigma'M_A)'(\Sigma'(M_{\varphi})U_xU_{x\Psi}}{U_{xx}} - \frac{\Phi(\Sigma'M_A)'(\Sigma'(M_{\Gamma}))U_xU_{x\Phi}}{U_{xx}} \\ &-\frac{\Psi\Phi(\Sigma'(M_{\varphi}))'(\Sigma'(M_{\Gamma}))U_{x\Phi}U_{x\Psi}}{U_{xx}} + \frac{\Psi^2(\Sigma'(M_{\varphi}))'(\Sigma'(M_{\varphi}))U_{x\Psi}^2}{2U_{xx}} + \\ &\frac{\Phi^2(\Sigma'(M_{\Gamma}))'(\Sigma'(M_{\Gamma}))U_{x\Phi}^2}{2U_{xx}} - \frac{\Psi(\Sigma'M_A)'\sigma'_{\varphi}U_xU_{x\Psi}}{U_{xx}} \\ &-\frac{\Psi^2(\Sigma'(M_{\varphi})'\sigma'_{\varphi}U_{x\Psi}^2}{U_{xx}} + \frac{\Psi\Phi(\Sigma'(M_{\Gamma})'\sigma'_{\varphi}U_{x\Psi}U_{x\Phi}}{U_{xx}} - \frac{\Phi(\Sigma'M_A)'\sigma'_{\Gamma}U_xU_{x\Phi}}{U_{xx}} \\ &+\frac{\Phi^2(\Sigma'(M_{\Gamma})'\sigma'_{\Gamma}U_{x\Phi}^2}{U_{xx}} - \frac{\Psi\Phi(\Sigma'(M_{\varphi})'\sigma'_{\Gamma}U_{x\Psi}U_{x\Phi}}{U_{xx}} \\ &+\frac{1}{2}\Psi^2\sigma_{\varphi}\sigma'_{\varphi}U_{\Psi\Psi} - \Psi\Phi\sigma_{\varphi}\sigma'_{\Gamma}U_{\Psi\Phi} + \frac{1}{2}\Phi^2\sigma_{\Gamma}\sigma'_{\Gamma}U_{\Phi\Phi} = 0, \end{aligned} \tag{26}$$

where $M_A = (\Sigma\Sigma')^{-1}A$, $M_{\varphi} = \Sigma^{-1}\sigma'_{\varphi}$ and $M_{\Gamma} = \Sigma^{-1}\sigma'_{\Gamma}$.

Let $U(t,v) = \dfrac{(vQ(t))^\gamma}{\gamma}$, so that $U_t = v^\gamma Q(t)^{\gamma-1}Q'(t)$, $U_x = v^{\gamma-1}Q(t)^\gamma$, $U_\Psi = v^{\gamma-1}Q(t)^\gamma$, $U_\Phi = -v^{\gamma-1}Q(t)^\gamma$, $U_{xx} = (\gamma-1)v^{\gamma-2}Q(t)^\gamma$, $U_{x\Psi} = (\gamma-1)v^{\gamma-2}Q(t)^\gamma$, $U_{x\Phi} = -(\gamma-1)v^{\gamma-2}Q(t)^\gamma$, $U_{\Psi\Psi} = (\gamma-1)v^{\gamma-2}Q(t)^\gamma$, $U_{\Phi\Phi} = (\gamma-1)v^{\gamma-2}Q(t)^\gamma$. Substituting the above partial derivatives into (26), we have

$$\begin{aligned}
&Q(t)^{-1}Q'(t) + rxv^{-1} + r\Psi v^{-1} + \alpha\Phi v^{-1} + \Psi\sigma_\varphi\theta v^{-1} - \frac{(M_A)'A}{\gamma-1}\\
&-\Psi(M_\varphi)'Av^{-1} - \Phi(M_\Gamma)'Av^{-1} + \frac{(\Sigma'M_A)'(\Sigma'M_A)}{2(\gamma-1)}\\
&+\Psi(\Sigma'M_A)'(\Sigma'(M_\varphi)v^{-1} + \Phi(\Sigma'M_A)'(\Sigma'(M_\Gamma))v^{-1}\\
&+\Psi\Phi(\Sigma'(M_\varphi))'(\Sigma'(M_\Gamma))(\gamma-1)v^{-2} + \frac{\Psi^2(\Sigma'(M_\varphi))'(\Sigma'(M_\varphi))(\gamma-1)v^{-2}}{2}\\
&+\frac{\Phi^2(\Sigma'(M_\Gamma))'(\Sigma'(M_\Gamma))(\gamma-1)v^{-2}}{2} - \Psi(\Sigma'M_A)'\sigma_\varphi' v^{-1}\\
&-\Psi^2(\Sigma'(M_\varphi)')\sigma_\varphi'(\gamma-1)v^{-2} + \Psi\Phi(\Sigma'(M_\Gamma)')\sigma_\varphi'(\gamma-1)v^{-2} + \Phi(\Sigma'M_A)'\sigma_\Gamma' v^{-1}\\
&+\Phi^2(\Sigma'(M_\Gamma)')\sigma_\Gamma'(\gamma-1)v^{-2} - \Psi\Phi(\Sigma'(M_\varphi)')\sigma_\Gamma'(\gamma-1)v^{-2}\\
&+\frac{1}{2}\Psi^2\sigma_\varphi\sigma_\varphi'(\gamma-1)v^{-2} + \Psi\Phi\sigma_\varphi\sigma_\Gamma'(\gamma-1)v^{-2} + \frac{1}{2}\Phi^2\sigma_\Gamma\sigma_\Gamma'(\gamma-1)v^{-2} = 0.
\end{aligned} \tag{27}$$

The solution to (27) can be obtained numerically. Using the above partial derivatives on (25), we have

$$\begin{aligned}
(\Delta'(t))^* = \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} + \frac{\Psi(t)}{X^*(t)}\left(\frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} - \Sigma^{-1}\sigma_\varphi'\right)\\
- \frac{\Phi(t)}{X^*(t)}\left(\frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} + \Sigma^{-1}\sigma_\Gamma'\right).
\end{aligned} \tag{28}$$

Suppose that σ_φ is a zero vector, we have

$$(\Delta_1'(t))^* = \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} + \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma}\frac{\Psi(t)}{X^*(t)} - \frac{\Phi(t)}{X^*(t)}\left(\frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} + \Sigma^{-1}\sigma_\Gamma'\right). \tag{29}$$

Suppose also that σ_Γ is a zero vector, we have

$$(\Delta_2'(t))^* = \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma} + \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma}\frac{\Psi(t)}{X^*(t)} - \frac{(\Sigma\Sigma')^{-1}A}{1-\gamma}\frac{\Phi(t)}{X^*(t)}. \tag{30}$$

We observe from (30) that if the risks associated to stochastic cash inflows and stochastic cash outflows can be eliminated, the optimal portfolio value for the investor will increase over time. All the figures were obtain by setting $x_0 = 10$, $\sigma_I = (0.2, 0)$, $r = 0.04$, $\theta^I = 0.125$, $\mu = 0.09$, $\sigma^S = (0.3, 0.4)$, $\sigma_\varphi = (0.18, 0.24)$,

$\sigma_\Gamma = (0.15, 0.22)$, $T = 30$, $\gamma = 0.3$, $\omega = 0.0292$, $\delta = 0.02$, $\varphi_0 = 0.02$, $\Gamma_0 = 0.005$. Figure 1 and Figure 2 show the variational optimum portfolio value in stock and index bond, respectively up to terminal time. We observe the portfolios are less volatile between initial period up to about 15 years and highly volatile thereafter. Figure 3 shows the portfolio value in cash account up to final time. The variational behaviour of these three portfolios was due to stochastic nature of the cash flows. Figure 4 – Figure 6 show the portfolio values of the investment profile when the cash inflows and cash outflows are deterministic, i.e., when $\sigma_\varphi = (0, 0)$, $\sigma_\Gamma = (0, 0)$. In Figures 4 and 5 we found that investment portfolio in stock and index bond, respectively, remain nonnegative from the first year to about 20 years and then declined gradually. In Figure 6, we found the from the first year to about 20 years the portfolio value in cash account remain negative and nonnegative thereafter. These show that the fund should be invested in stock and index bond for the first 20 years and then gradually shift the fund to cash account thereafter. This is a suitable portfolio risk hedging strategy for a long term investment.

In the next section, we consider the optimal net investment wealth for the

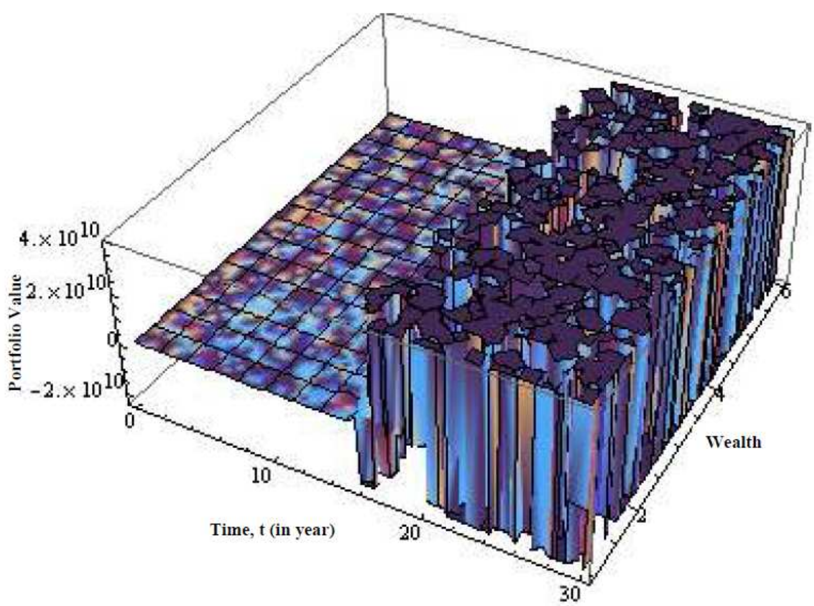

Figure 1: Optimum Portfolio Value in Stock (where the cash flows are stochastic)

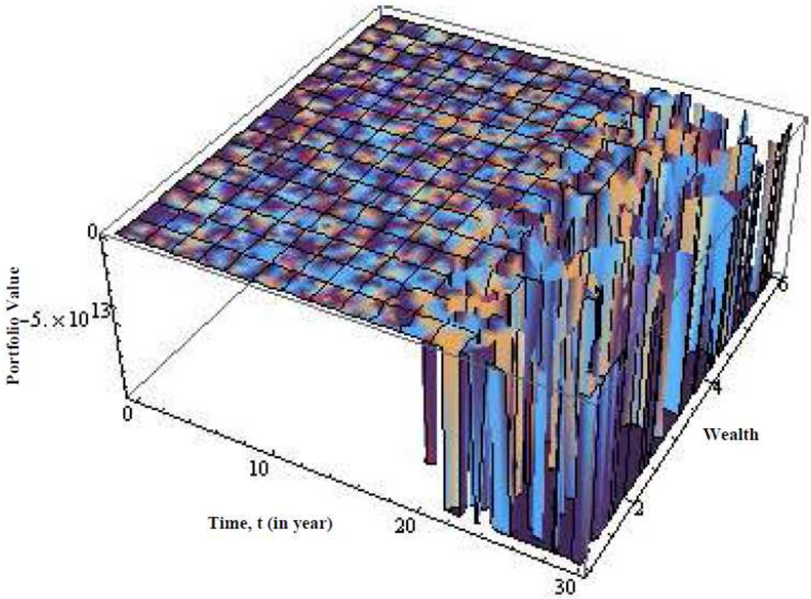

Figure 2: Optimum Portfolio Value in index bond (where the cash flows are stochastic)

investor.

7. Optimal Net Wealth for the Investor

In this section, we consider the optimal net investment wealth of the investor at time t.

Proposition 2. *Suppose that the net investment wealth dynamic satisfies (21) with (28), then the optimal net investment wealth dynamic of $V^*(t)$ is*

$$\begin{aligned}
dV^*(t) &= (rX^*(t) + \frac{A'[(\Sigma\Sigma')^{-1}]'AX^*(t)}{1-\gamma})dt + \frac{(\Sigma^{-1}A)'X^*(t)}{1-\gamma}dW(t) \\
&+\Psi(t)((r+\sigma_\varphi\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A)dt + (\frac{[\Sigma^{-1}A]'}{1-\gamma})dW(t)) \qquad (31) \\
&-\Phi(t)((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A)dt + (\frac{(\Sigma^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)dW(t).
\end{aligned}$$

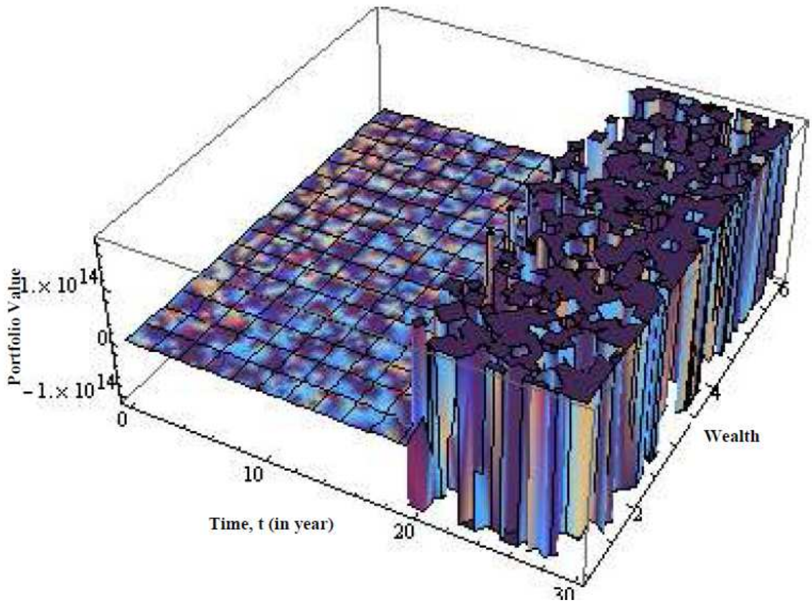

Figure 3: Optimum Portfolio Value in Cash Account (where the cash flows are stochastic)

Proof. The result (net investment wealth process of the investor) follows by substituting (28) into (21) and then simplifying.

We therefore deduce from (31) that

$$dX^*(t) = (rX^*(t) + \frac{A'[(\Sigma\Sigma')^{-1}]'AX^*(t)}{1-\gamma})dt + \frac{(\Sigma^{-1}A)'X^*(t)}{1-\gamma}dW(t); \quad (32)$$

$$d\Psi(t) = \Psi(t)((r+\sigma_\varphi\theta+\frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma}-[\Sigma^{-1}\sigma'_\varphi]'A)dt+(\frac{[\Sigma^{-1}A]'}{1-\gamma})dW(t)); \quad (33)$$

$$d\Phi(t) = \Phi(t)((\alpha+\frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma}+[\Sigma^{-1}\sigma'_\Gamma]'A)dt+(\frac{(\Sigma^{-1}A)'}{1-\gamma}+2\sigma_\Gamma)dW(t). \quad (34)$$

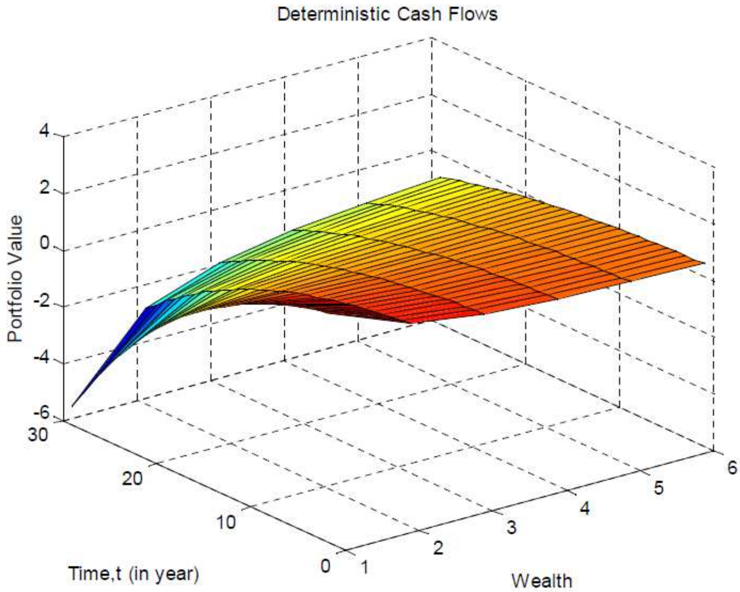

Figure 4: Optimum Portfolio Value in Stock

Therefore,

$$
\begin{aligned}
V^*(t) &= x_0 \exp[(r + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'(\Sigma^{-1}A)}{(1-\gamma)^2}\right))t \\
&+ \frac{(\Sigma^{-1}A)'}{1-\gamma}W(t)] + \Psi_0 \exp[((r + \sigma_\varphi\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A \\
&- \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'(\Sigma^{-1}A)}{(1-\gamma)^2}\right))t + \frac{[\Sigma^{-1}A]'}{1-\gamma}W(t))] \\
&- \Phi_0 \exp[((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A \\
&- \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'\Sigma^{-1}A)}{(1-\gamma)^2} + 2\frac{(\Sigma^{-1}A)'\sigma'_\Gamma}{1-\gamma} + 4\sigma_\Gamma\sigma'_\Gamma)\right))t + (\frac{(\Sigma^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)W(t)].
\end{aligned} \tag{35}
$$

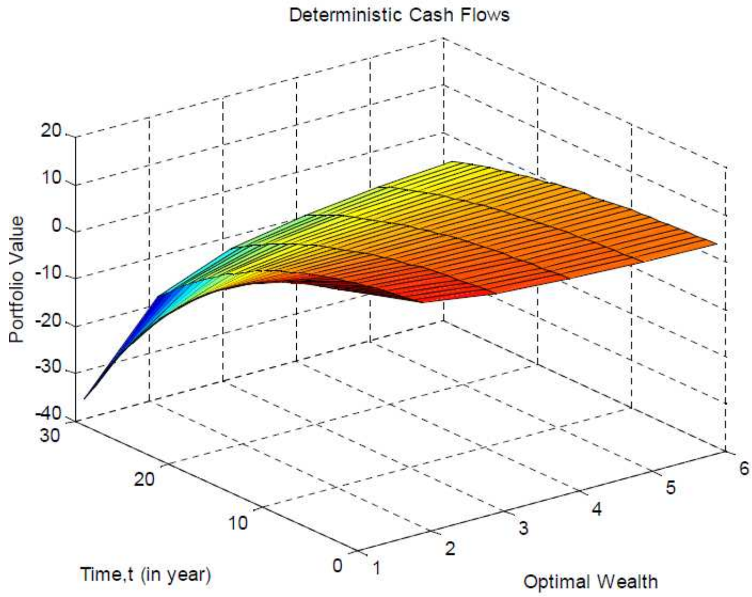

Figure 5: Optimum Portfolio Value in Index Bond

The terminal investor's optimal wealth is obtain as follows

$$
\begin{aligned}
V^*(T) &= x_0 \exp[(r + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'(\Sigma^{-1}A)}{(1-\gamma)^2}\right))T \\
&+ \frac{(\Sigma^{-1}A)'}{1-\gamma}W(T)] + \Psi_0 \exp[((r + \sigma_\varphi\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A \\
&- \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'(\Sigma^{-1}A)}{(1-\gamma)^2}\right))T + \frac{[\Sigma^{-1}A]'}{1-\gamma}W(T))] \\
&- \Phi_0 \exp[((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A \\
&- \frac{1}{2}\left(\frac{(\Sigma^{-1}A)'\Sigma^{-1}A)}{(1-\gamma)^2} + 2\frac{(\Sigma^{-1}A)'\sigma'_\Gamma}{1-\gamma} + 4\sigma_\Gamma\sigma'_\Gamma)\right))T + (\frac{(\Sigma^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)W(T)].
\end{aligned} \tag{36}
$$

Taking mathematical expectation of (32), (33) and (34) and then solving the

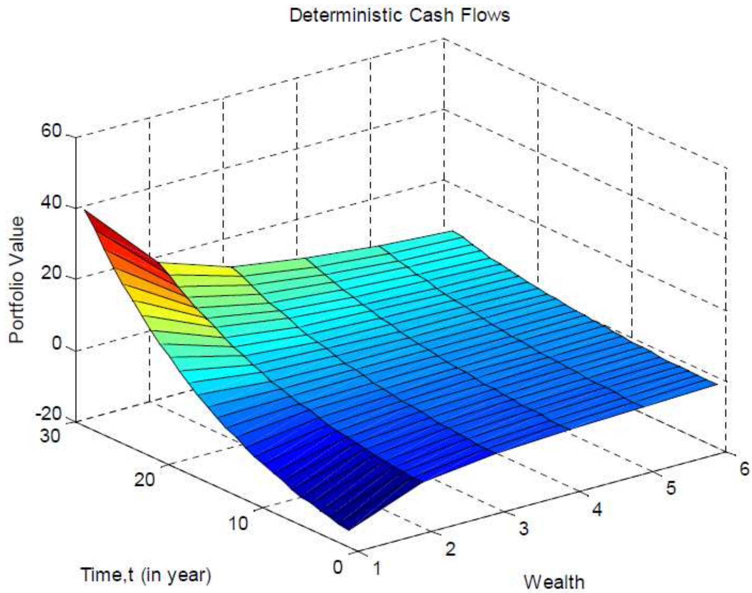

Figure 6: Optimum Portfolio Value in Cash Account

resulting ordinary differential equations, we have the following:

$$\left.\begin{array}{l} E(X^*(t)) = x_0 \exp((r+y)t) \\ E(\Psi(t)) = \Psi_0 \exp((r+y+\sigma_\varphi\theta - [\Sigma^{-1}\sigma'_\varphi]'A)t) \\ E(\Phi(t)) = \Phi_0 \exp\left[\left(\alpha + y + [\Sigma^{-1}\sigma'_\Gamma]'A - (\dfrac{(\Sigma^{-1}A)'\sigma'_\Gamma}{1-\gamma} + \sigma_\Gamma\sigma'_\Gamma)\right)t\right] \end{array}\right\}, \tag{37}$$

where $y = \dfrac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma}$.

Using the Ito lemma and taking the mathematical expectation on (32), (33) and (34) and solving the resulting ordinary differential equations, we have their second moments to be

$$\left.\begin{array}{l} E(X^{*2}(t)) = x_0^2 \exp((2r+2y+zz')t) \\ E(\Psi^2(t)) = \Psi_0^2 \exp((2r+2y+2\sigma_\varphi\theta - 2[\Sigma^{-1}\sigma'_\varphi]'A + zz')t) \\ E(\Phi^2(t)) = \Phi_0^2 \exp((2\alpha+2y+2[\Sigma^{-1}\sigma'_\Gamma]'A - 2(\dfrac{(\Sigma^{-1}A)'\sigma'_\Gamma}{1-\gamma} + \sigma_\Gamma\sigma'_\Gamma) + zz'_1)t), \end{array}\right\}, \tag{38}$$

where $z = \dfrac{(\Sigma^{-1}A)'}{1-\gamma}$ and $z_1 = z + 2\sigma_\Gamma$.

Proposition 3. *Suppose that $V^*(T)$ satisfies (36), then*

$$
\begin{aligned}
&Var(V^*(T)) = (E(X^*(T)))^2(\exp[zz'T]-1)\\
&+(E(\Psi(T)))^2(\exp[zz'T]-1)+(E(\Phi(T)))^2(\exp[z_1z_1'T]-1)\\
&+2E(X^*(T))E(\Psi(T))(\exp[zz'T]-1)-2E(X^*(T))E(\Phi(T))(\exp[zz_1'T]-1)\\
&-2E(\Psi(T)E(\Phi(T)(\exp[zz_1'T]-1).
\end{aligned}
\tag{39}
$$

Proof. First, we take the mathematical expectation of (36). It then follows that the expected terminal net investment wealth is

$$
\begin{aligned}
&E(V^*(T)) = x_0\exp[(r+y)T]\\
&+\Psi_0\exp[(r+\sigma_\varphi\theta+y-[\Sigma^{-1}\sigma_\varphi']'A)T]\\
&-\Phi_0\exp[(\alpha+y+[\Sigma^{-1}\sigma_\Gamma']'A-(\frac{(\Sigma^{-1}A)'\sigma_\Gamma'}{1-\gamma}+\sigma_\Gamma\sigma_\Gamma'))T],
\end{aligned}
\tag{40}
$$

and (40) is the expected terminal wealth for the investor. Observe that it is a function of the wealth process, cash inflows and cash outflows at the terminal time.

Using the Ito lemma and taking the mathematical expectation of (31) and then solve, we have the second moment of $V^*(T)$ to be

$$
\begin{aligned}
&E(V^*(T))^2 = x_0^2\exp[(2(r+y)+zz')T]\\
&+\Psi_0^2\exp[(2(r+\sigma_\varphi\theta+y-[\Sigma^{-1}\sigma_\varphi']'A)+zz')T]\\
&+\Phi_0^2\exp[(2(\alpha+y+[\Sigma^{-1}\sigma_\Gamma']'A-(\frac{(\Sigma^{-1}A)'\sigma_\Gamma'}{1-\gamma}+\sigma_\Gamma\sigma_\Gamma'))+z_1z_1')T]\\
&+2x_0\Psi_0\exp[(2r+2y+\sigma_\varphi\theta-(\Sigma^{-1}\sigma_\varphi')'A+zz')T]\\
&-2x_0\Phi_0\exp[(r+\alpha+2y+(\Sigma^{-1}\sigma_\Gamma')'A-(\frac{(\Sigma^{-1}A)'\sigma_\Gamma'}{1-\gamma}+\sigma_\Gamma\sigma_\Gamma')+zz_1')T]-\\
&2\Psi_0\Phi_0\exp[(r+\alpha+\sigma_\varphi\theta+2y+(\Sigma^{-1}\sigma_\Gamma')'A-(\Sigma^{-1}\sigma_\varphi')'A\\
&-(\frac{(\Sigma^{-1}A)'\sigma_\Gamma'}{1-\gamma}+\sigma_\Gamma\sigma_\Gamma')+zz_1')T].
\end{aligned}
\tag{41}
$$

It then follows that

$$
\begin{aligned}
&E(V^*(T))^2 = (E(X^*(T)))^2\exp[zz'T]+(E(\Psi(T)))^2\exp[zz'T]\\
&+(E(\Phi(T)))^2\exp[z_1z_1'T]+2E(X^*(T))E(\Psi(T))\exp[zz'T]\\
&-2E(X^*(T))E(\Phi(T))\exp[zz_1'T]-2E(\Psi(T)E(\Phi(T)\exp[zz_1'T].
\end{aligned}
\tag{42}
$$

The variance of the net investment wealth of the investor is given as

$$\begin{aligned}&Var(V^*(T)) = E(V^*(T))^2 - (E(V^*(T)))^2 = (E(X^*(T)))^2(\exp[zz'T]-1)\\&+(E(\Psi(T)))^2(\exp[zz'T]-1) + (E(\Phi(T)))^2(\exp[z_1z_1'T]-1)\\&+2E(X^*(T))E(\Psi(T))(\exp[zz'T]-1) - 2E(X^*(T))E(\Phi(T))(\exp[zz_1'T]-1)\\&-2E(\Psi(T)E(\Phi(T)(\exp[zz_1'T]-1).\end{aligned} \tag{43}$$

Proposition 4. *$z_1 = z$ if and only if σ_Γ is a zero vector.*

Proposition 5. *(Market efficiency test) Suppose that Proposition 4 holds, then there exists a function $K(T)$ such that*

$$\begin{aligned}&Var(V^*(T)) = K(T)(E(V^*(T)))^2,\\&\text{where } K(T) = e^{zz'T}-1.\end{aligned} \tag{44}$$

Proof. Setting $z_1 = z$ in (43) and then simplify, the result follows. □

Proposition 5 tells us that market efficiency test can be obtained if the risks associating with the cash outflows are eliminated. The following proposition characterizes the efficient frontier in terms of the expected returns and variance.

Proposition 6. (Efficient Frontier) *Suppose that Proposition 5 holds, then*

$$E(V^*(T)) = \sigma(V^*(T))\frac{1}{\sqrt{e^{zz'T}-1}}, \quad \text{for } T > 0. \tag{45}$$

Equation (45) in Proposition 6 tells us that the expected terminal wealth can be expressed as a function of its standard deviation. Therefore, the efficient frontier of portfolios in the mean-standard deviation diagram is a straight line with gradient $\frac{1}{\sqrt{e^{zz'T}-1}}$ known as the price of risk. It measures the increase of the mean of the terminal wealth if the standard deviation of the terminal wealth increases by one unit.

Proposition 7. *Suppose that Σ is a zero matrix, z and z_1 are zero vectors, then the variance of the wealth for the investor, $Var(V(T))$ must be zero, i.e.,*

$$E(V^{*2}(T)) = [E(V^*(T))]^2 = [(x_0+\Psi_0)e^{rT} - \Phi_0 e^{\alpha T}]^2. \tag{46}$$

It then follows that

$$E(V^*(T)) = (x_0+\Psi_0)e^{rT} - \Phi_0 e^{\alpha T}. \tag{47}$$

Formula (47) gives the net investment wealth for the investor at terminal time, T. The amount $(x_0 + \Psi_0)e^{rT}$ is invested only in cash account and the amount $\Phi_0 e^{\alpha T}$ is the value of the cash outflows at time T. Observe that the investment will be in the money if and only if $\Phi_0 < (x_0 + \Psi_0)e^{(r-\alpha)T}$, out of the money if and only if $\Phi_0 > (x_0 + \Psi_0)e^{(r-\alpha)T}$ and at the money if and only if $\Phi_0 = (x_0 + \Psi_0)e^{(r-\alpha)T}$. Observe that if we set $\alpha = r$, (46) becomes

$$E(V^{*2}(T)) = [E(V^*(T))]^2 = v_0^2 e^{2rT}. \tag{48}$$

It implies that $E(V^*(T)) = v_0 e^{rT}$. This tells us that for zero variance to occur, the entire portfolio must remain only in cash account. From Proposition 7 at zero variance, suppose that the investment only involve stochastic cash inflows (i.e., $\Gamma_0 = 0$, which implies that $\Phi_0 = 0$), then the expected net investment wealth for the investor is obtained as $E(V^*(T)) = (x_0 + \Psi_0)e^{rT}$. This tells us that the initial endowment, x_0 and the present value of the cash inflows, Ψ_0 should remain only in cash account.

If also the investment only involved stochastic cash outflows (i.e., $\varphi_0 = 0$ which implies that $\Psi_0 = 0$), we have that the expected net investment wealth for the investor is $E(V^*(T)) = x_0 e^{rT} - \Phi_0 e^{\alpha T}$. It follows that $E(V^*(T)) = 0$ if $x_0 e^{rT} = \Phi_0 e^{\alpha T}$, it implies that for $1 - e^{-\beta T} > e^{-\beta T}$, we have $T < T^* = \frac{1}{r-\alpha-\beta}\log_e\left[\frac{\beta x_0}{\Gamma_0}\right]$. This specifies explicitly the critical length of time horizon, $T^* = \frac{1}{r-\alpha-\beta}\log_e\left[\frac{\beta x_0}{\Gamma_0}\right]$ were the market portfolio value will be equals the discounted cash outflows. Interestingly, T^* at $E(V^*(T)) = 0$ depends only on the growth of the cash outflows $r - \alpha - \beta$, initial endowment, x_0 and initial cash outflows, Γ_0.

At equilibrium, we assume that the discounted cash inflows and discounted cash outflows will be adjusted to depends on the optimal wealth. In that case, we will observe that the discounted cash flows can be expressed as a function of optimal wealth process. If the target of the investor is known, we can determine the discounted cash inflows and the cash outflows at any time t. Observe from (35) that $\Psi(t)$ and $\Phi(t)$ can be re-expressed in terms of $X^*(t)$ as follows:

$$\Psi(t) = \frac{\Psi_0 X^*(t)}{x_0}\exp\left[\left(\sigma_\varphi\theta - [(\Sigma)^{-1}\sigma_\varphi']'A\right)t\right], \tag{49}$$

$$\Phi(t) = \frac{\Phi_0 X^*(t)}{x_0}\exp\Big[\Big(\alpha - r + [(\Sigma)^{-1}\sigma_\Gamma']'A - \big(\frac{((\Sigma)^{-1}A)'\sigma_\Gamma'}{1-\gamma} + 2\sigma_\Gamma\sigma_\Gamma'\big)\Big)t + 2\sigma_\Gamma W(t)\Big]. \tag{50}$$

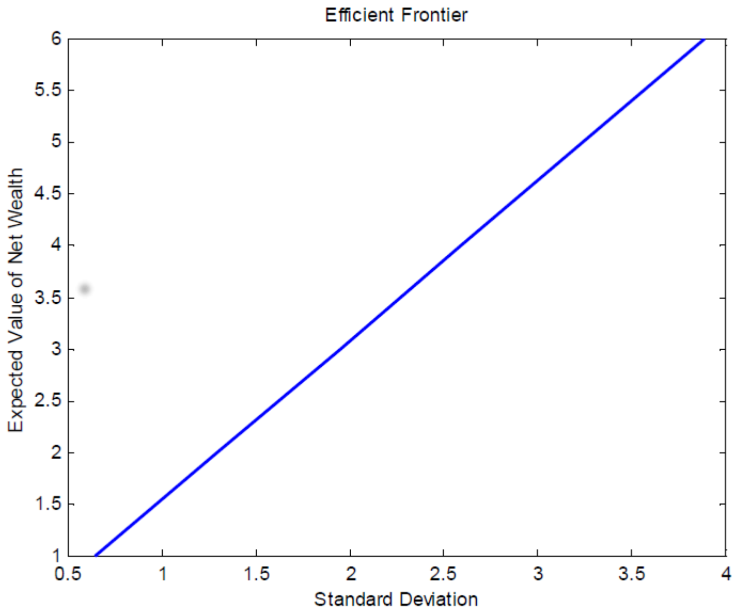

Figure 7: The Efficient Frontier

These show that the values of the discounted cash inflows and the outflows ultimately depend on the optimal wealth process and the investor's risk preference over time.

Figure 7 depicts the efficient frontier for the three classes of assets in mean-standard deviation. It shows that for the expected net investment wealth of 6 million stand the risk of losing 3.8 million. Figure 8 gives the discounted cash inflows against optimal wealth over a period of 30 years. It shows that as the optimal wealth increases the value of the discounted cash inflows increases a long side over time. Figure 9 shows the expected discounted cash outflows against optimal wealth. We observe that as optimal wealth increases the expected discounted cash outflows increases rapidly over time.

8. Conclusion

In this paper we have studied the optimal net investment wealth with discounted stochastic cash flows for an investor. The dynamics of the net investment wealth process involving two risky assets and a cash account has been established. Market efficiency test and efficient frontier for the three classes of assets are

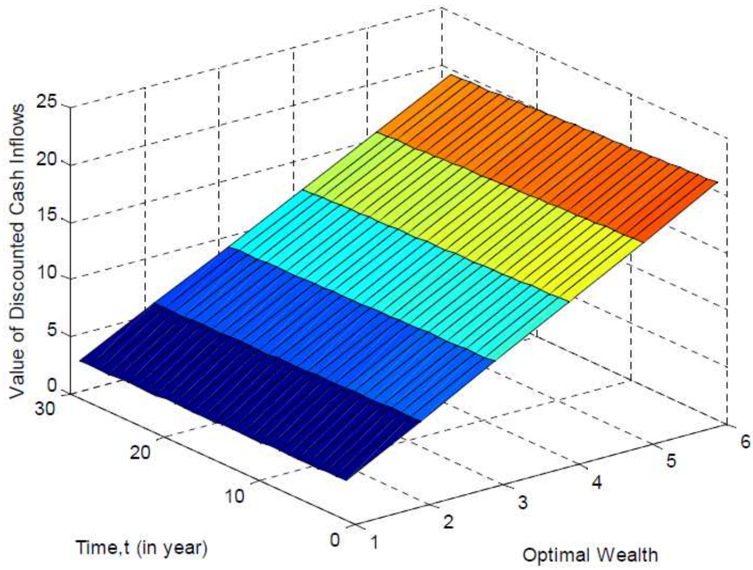

Figure 8: Value of Discounted Cash Inflows versus Optimal Wealth overtime

given. We obtain the optimal terminal net investment wealth and show that the discounted cash inflows and cash outflows depend on the optimal wealth of the investor. The expected terminal net investment wealth with zero variance is also established.

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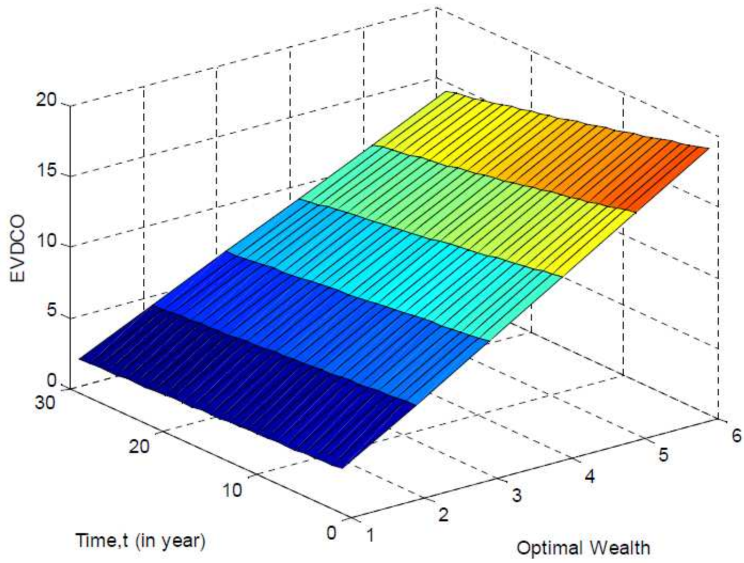

Figure 9: Expected Value of Discounted Cash Outflows (EVDCO) versus Optimal Wealth overtime

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