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ON THE LOCAL WELL-POSEDNESS OF A BIDIMENSIONAL VERSION OF THE BENJAMIN-ONO EQUATION

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Abstract: In this paper we show that the Cauchy problem

$$\begin{cases} u_t + \mathcal{H}^{(y)}\partial_x^2 u + u^p u_x = 0, \quad p \in \mathbb{N}, \\ u(0) = \phi(x,y) \end{cases}$$

is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$, for $s > 2$ and that as in the case of the BO (Benjamin-Ono) equation, there is a lack of persistence in X^s.

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1. Introduction

The purpose of this paper is to show that the Cauchy problem for

$$u_t + \mathcal{H}^{(y)}\partial_x^2 u + u^p u_x = 0, \tag{1}$$

is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$, for $s > 2$. Observe that (1)

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is a modification of the Benjamin-Ono equation

$$\partial_t u + \mathcal{H}^{(x)} \partial_x^2 u + u \partial_x u = 0, \tag{2}$$

which describes certain models in physics related to wave propagation in a stratified thin regions (see [3]). This last equation shares with the KdV equation

$$u_t + u_x + u u_x + u_{xxx} = 0 \tag{3}$$

many interesting properties. For example, both equations possess infinite conservation laws, they have solitary waves as solutions which are stable and behave like soliton (this last is evidenced by the existence of multisoliton type solutions) (see [2] and [13]). Also, the local and global well-posedness were proven in the Sobolev spaces context (in low regularity spaces inclusive, see, e.g., [7], [14], [11], [12] and [15]).

The plan of this paper is the following: In Section 2, we present the basic notations and results that are needed. In Section 3, we examine the local well-posedness in H^s and Section 4, using some ideas from [10], we show that there is not persistence of 1 in X^s.

The main tool that we use is the abstract theory developed by Kato in [8] to prove the local well-posedness of quasi-linear equations of evolution. Kato considered the problem

$$\begin{aligned} &\partial_t u + A(t,u)u = f(t,u) \in X,\ 0 < t, \\ &u(0) = u_0 \in Y, \end{aligned} \tag{4}$$

in a Banach space X with initial data in a dense subspace Y of X, where A is a map from $\mathbb{R} \times X$ into the linear operators of X with dense domain and $f(t,u)$ is a function from $\mathbb{R} \times Y$ to X, which satisfy the following conditions:

(X) There exists an isometric isomorphism S from Y to X.

There exists $T_0 > 0$ and W a open ball with center w_0 such that:

(A_1) For each $(t,y) \in [0,T_0] \times W$, the linear operator $A(t,y)$ belongs to $G(X,1,\beta)$, where β is a positive real number. In other words, $-A(t,y)$ generates a C_0 semigroup such that

$$\|e^{-sA(t,y)}\|_{\mathcal{B}(X)} \le e^{\beta s}, \quad s \in [0,\infty).$$

It should be noted that if X is a Hilbert space, $A \in G(X,1,\beta)$ if and only if,

a) $\langle Ay, y\rangle_X \ge -\beta \|y\|_X^2$ for all $y \in D(A)$,

b) $(A + \lambda)$ is onto for all $\lambda > \beta$.

(A_2) For all $(t,y) \in [0,T_0] \times W$ the operator

$$B(t,y) = [S, A(t,y)]S^{-1} \in \mathcal{B}(X)$$

and is uniformly bounded, i.e., there exists $\lambda_1 > 0$ such that

$$\|B(t,y)\|_{\mathcal{B}(X)} \leq \lambda_1 \text{ for all } (t,y) \in [0,T_0] \times W.$$

In addition, for some $\mu_1 > 0$, for all y and $z \in W$,

$$\|B(t,y) - B(t,z)\|_{\mathcal{B}(X)} \leq \mu_1 \|y - z\|_Y.$$

(A_3) $Y \subseteq D(A(t,y))$, for each $(t,y) \in [0,T_0] \times W$, (the restriction of $A(t,y)$ to Y belonging to $\mathcal{B}(Y,X)$) and, for each fixed $y \in W$, $t \to A(t,y)$ is strongly continuous. Furthermore, for each fixed $t \in [0,T_0]$, it is satisfied the following Lipschitz condition:

$$\|A(t,y) - A(t,z)\|_{\mathcal{B}(Y,X)} \leq \mu_2 \|y - z\|_X,$$

where $\mu_2 \geq 0$ is a constant.

(A_4) $A(t,y)w_0 \in Y$ for all $(t,y) \in [0,T] \times W$. Also, there exists a constant λ_2 such that

$$\|A(t,y)w_0\|_Y \leq \lambda_2, \text{ for all } (t,y) \in [0,T_0] \times W.$$

(f_1) f is a bounded function from $[0,T_0] \times W$ in Y, i.e., there exists λ_3 such that

$$\|f(t,y)\|_Y \leq \lambda_3, \text{ for all } (t,y) \in [0,T_0] \times W,$$

Besides, the function $t \in [0,T_0] \mapsto f(t,y) \in Y$ is continuous with respect to X topology and, for all y and $z \in Y$, we have that

$$\|f(t,y) - f(t,z)\|_X \leq \mu_3 \|y - z\|_X,$$

when $\mu_3 \geq 0$ is a constant.

Theorem 1 (Kato). *Suppose that the conditions (X), $(A_1) - (A_4)$ y (f_1) are satisfied. For $u_0 \in Y$, there exist $0 < T < T_0$ and a unique $u \in C([0,T];Y) \cap C^1((0,T);X)$ solution to (4). Besides, the map $u_0 \to u$ is continuous in the following sense: consider the following sequence of Cauchy problems,*

$$\begin{aligned} &\partial_t u_n + A_n(t,u_n)u_n = f_n(t,u_n) \qquad t > 0 \\ &u_n(0) = u_{n_0} \qquad\qquad\qquad\qquad\quad n \in \mathbb{N}. \end{aligned} \tag{5}$$

Assume that conditions (X), (A_1)–(A_4) *and* (f_1) *hold for all* $n \geq 0$ *in* (5), *with the same* X, Y *and* S, *and the corresponding* β, λ_1–λ_3, μ_2–μ_3 *can be chosen independently from* n. *Also assume that*

$$\begin{aligned} s\text{-}\lim_{n\to\infty} A_n(t,w) &= A(t,w) \text{ in } B(X,Y) \\ s\text{-}\lim_{n\to\infty} B_n(t,w) &= B(t,w) \text{ in } B(X) \\ \lim_{n\to\infty} f_n(t,w) &= f(t,w) \text{ in } Y \\ \lim_{n\to\infty} u_{n_0} &= u_0 \text{ in } Y, \end{aligned}$$

where s-lim *denotes the strong limit. Then,* T *can be chosen in such a way that* $u_n \in C([0,T],Y) \cap C^1((0,T),X)$ *and*

$$\lim_{n\to\infty} \sup_{[0,T]} \|u_n(t) - u(t)\|_Y = 0.$$

A proof of this theorem can be seen in [8].

2. Preliminaries

The following notations will be used through this paper.

1. $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space.
2. $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions.
3. For $f \in \mathcal{S}'(\mathbb{R}^2)$, $\widehat{f}$ is the Fourier transform of f and $\check{f}$ is the inverse Fourier transform of f. We recall that

   $$\widehat{f}(\xi,\eta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x,y)e^{-i(x\xi+y\eta)}\,dxdy,$$

   for all $(\xi,\eta) \in \mathbb{R}^2$, when $f \in \mathcal{S}(\mathbb{R}^2)$.
4. $\mathcal{H}^{(y)}$ is the Hilbert transform with respect to the variable y. If $f \in \mathcal{S}(\mathbb{R}^2)$,

   $$\mathcal{H}^{(y)} f(x,y) = \sqrt{\frac{2}{\pi}}\left(\text{p.v.}\int_{-\infty}^{\infty} \frac{1}{\xi - y} f(x,\xi)\,d\xi\right).$$
5. For $s \in \mathbb{R}$, $H^s = H^s(\mathbb{R}^2)$ is the Sobolev space of order s.

6. The inner product in H^s is denoted as

$$\langle f, g\rangle_s = \int_{\mathbb{R}^2} (1+\xi^2+\eta^2)^s \widehat{f}(\xi,\eta)\overline{\widehat{g}(\xi,\eta)}d\xi d\eta.$$

7. $\Lambda^s = (1-\Delta)^{s/2}$.

8. If X, Y are Banach spaces, $B(X,Y)$ is the space of all continuous linear operators endowed with the norm:

$$\|T\|_{B(X,Y)} = \sup_{\|x\|=1} \|Tx\|.$$

   If $X = Y$ we simply write $B(X)$.

9. $[A, B]$ will denote the commutator of A and B.

10. $X^s = H^s(\mathbb{R}^2) \cap L^2(1+y^2)$ where $L^2(1+y^2)$ is the space of all real valued measurable functions such that

$$\|f\|_{L^2(1+y^2)} = \left(\int f^2(x,y)(1+y^2)\right)^{\frac{1}{2}} dxdy < \infty.$$

   X^s is a Hilbert space when provided with the inner product:

$$(f,g)_{X^s} = (f,g)_s + (f,g)_{L^2(1+y^2)}.$$

The following result about commutators of operators due to Kato is a part of the important stock of tools that are used in the analysis (its proof can be found in [9]).

Proposition 2 (Kato's inequality)**.** *Let* $f \in H^s$, $s > 2$, $\Lambda = (1-\Delta^2)^{1/2}$ *and* M_f *be the multiplication operator by* f*. Then, for* $|\tilde{t}|$, $|\tilde{s}| \leq s-1$, $\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}} \in B(L^2(\mathbb{R}^2))$ *and*

$$\left\|\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}}\right\|_{B(L^2(\mathbb{R}^2))} \leq c\|\nabla f\|_{H^{s-1}}. \tag{6}$$

2. Local Theory in Sobolev Spaces

In this section, we prove that the Cauchy problem

$$\begin{cases} u_t + \mathcal{H}^{(y)}\partial_x^2 u + u^p u_x = 0, \quad p \in \mathbb{N} \\ u(0) = \phi(x,y) \end{cases} \tag{7}$$

is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$, for $s > 2$.

Theorem 3. *Let $s > 2$ and $p \in \mathbb{N}$. For $\phi \in H^s(\mathbb{R}^2)$, there exist $T > 0$, that depends only on $\|\phi\|_s$, and a unique u belonging to $C([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-2}(\mathbb{R}^2))$ as solution to the Cauchy problem*

$$\begin{cases} u_t + \mathcal{H}^{(y)}\partial_x^2 u + u^p u_x = 0 \\ u(0) = \phi \end{cases}. \tag{8}$$

Furthermore, the map $\phi \mapsto u$ from H^s to $C([0,T], H^s)$ is continuous.

Proof. It is clear that u is a solution to (8) if and only if $v(t) = Q(t)u(t)$ is solution to

$$\begin{cases} \dfrac{dv}{dt} + A(t,v)v = 0 \\ v(0) = \phi \end{cases}, \tag{9}$$

where $Q(t) = e^{t\mathcal{H}^{(y)}\partial_{xx}}$ and

$$A(t,v) = Q(t)(Q(-t)v)^p \partial_x Q(-t).$$

Let us see for this problem that each one of the conditions of Kato's theorem (Theorem 1) is satisfied. For the moment, let $X = L^2(\mathbb{R}^2)$ and $Y = H^s(\mathbb{R}^2)$, for $s > 2$. It is clear that $S = (1-\Delta)^{\frac{s}{2}}$ is an isomorphism between X and Y. In the following lemmas we verify that the problem (9) satisfies the conditions (A_1)–(A_4) of Theorem 1.

Lemma 4. *$A(t,v) \in G(X, 1, \beta(v))$, where*

$$\beta(v) = \frac{1}{2}\sup_t \|\partial_x (Q(t)v)^p\|_{L^\infty(\mathbb{R}^2)}$$

(see the condition (A_1) before Theorem 1).

Proof. Since $\{Q(-t)\}$ is a strongly continuous group of unitary operators, and thanks to the observation immediately below of the condition (A_1) of Theorem 1, it follows the lemma. $\square$

Lemma 5. *If $S = (1-\Delta)^{s/2}$, then*

$$SA(t,v)S^{-1} = A(t,v) + B(t,v),$$

where $B(t,v)$ is a bounded operator in L^2, for all $t \in \mathbb{R}$ and $v \in H^s$, and satisfies the inequalities

$$\|B(t,v)\|_{B(X)} \leq \lambda(v) \tag{10}$$

$$\|B(t,v) - B(t,v')\|_{B(X)} \leq \mu(v,v')\|v' - v\|_s, \tag{11}$$

for all $t \in \mathbb{R}$, and every v and $v' \in H^s$, where

$$\lambda(v) = \sup_t C_s \|\nabla(Q(-t)v)^p\|_{s-1}$$

and $\mu(v,v') = C_{p,s}(\|v\|_s^{p-1} + \|v'\|_s^{p-1})$.

Proof. From Proposition 2 it follows that $[S,(Q(-t)v)^p]\partial_x S^{-1} \in B(X)$ and

$$\|[S,(Q(-t)v)^p]\partial_x S^{-1}\|_{B(X)} \leq C_s\|\nabla(Q(-t)v)^p\|_{s-1}.$$

Therefore, $B(t,v) \in B(X)$ and it satisfies (10).

Proceeding as above and taking into account that

$$\|v^p - w^p\|_s \leq C_{p,s}(\|u\|_s^{p-1} + \|v\|_s^{p-1})\|u - v\|_s, \tag{12}$$

for all u and $v \in H^s$, we can show (11). $\square$

Lemma 6. *$H^s(\mathbb{R}^2) \subset D(A(t,v))$ and $A(t,v)$ is a bounded operator from $Y = H^s(\mathbb{R}^2)$ to $X = L^2(\mathbb{R}^2)$ with*

$$\|A(t,v)\|_{B(X,Y)} \leq \lambda(v),$$

for all $v \in Y$, and where λ is as in Lemma 5. Also, the function $t \mapsto A(t,v)$ is strongly continuous from $\mathbb{R}$ to $B(Y,X)$, for all $v \in H^s$. Moreover, the function $v \mapsto A(t,v)$ satisfies the following Lipschitz condition

$$\|A(t,v) - A(t,v')\|_{B(Y,X)} \leq \mu(v,v')\|v - v'\|_X,$$

where μ is as in the lemma above.

Proof. In view of the fact that $Q(-t) = (e^{t\mathcal{H}^{(y)}\partial_{xx}})^{-1}$ is an unitary operator in $X = L^2(\mathbb{R}^2)$, from the definition of $A(t,v)$, it follows $H^s(\mathbb{R}^2) \subset D(A(t,v))$. In fact,

$$\begin{aligned} \|A(t,v)f\|_0 &= \|Q(-t)v)^p \partial_x Q(t)f\|_0 \\ &\le C_s \|(Q(-t)v)^p\|_s \|\partial_x f\|_0 \le \lambda(v)\|f\|_s, \end{aligned} \tag{13}$$

for all $f \in Y$.

Now, for all $t, t' \in \mathbb{R}$ and all $f, v \in Y$, we have

$$\begin{aligned} \|A(t,v)f - A(t',v)f\|_0 \le & \left\| \left(Q(t) - Q(t')\right) (Q(-t)v)^p \partial_x (Q(t)f) \right\|_0 \\ & + \left\| \left((Q(-t)v)^p - (Q(-t')v)^p\right) \partial_x (Q(t)f) \right\|_0 \\ & + \|(Q(-t')v)^p \partial_x (Q(t) - Q(t'))f\|_0. \end{aligned}$$

Since the group $\{Q(-t)\}_{t\in\mathbb{R}}$ is strongly continuous and the function $v \to v^p$ from Y itself is continuous, $t \mapsto A(t,v)$ is strongly continuous from $\mathbb{R}$ to $B(Y,X)$.

Finally, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} \|A(t,v')f - A(t,v)f\|_0 &\le \|(Q(t)v')^p - (Q(t)v)^p\|_0 \|\partial_x Q(t) f\|_\infty \\ &\le C_p(\|(Q(t)v)^{p-1}\|_\infty \\ &\qquad + \|(Q(t)v')^{p-1}\|_\infty)\|f\|_s \|v' - v\|_0 \\ &\le \mu(v,v')\|v' - v\|_0 \|f\|_s. \end{aligned}$$

This completes the proof of the lemma. □

The preceding lemmas show that the problem (9) satisfies the conditions of Theorem 1 and, therefore, for each $\phi \in H^s$, $s > 2$, there exists $T > 0$, which depends on $\|\phi\|_s$, and an unique $v \in C([0,T], H^s(\mathbb{R}^2) \bigcap C^1([0,T],\ H^{s-1}(\mathbb{R}^2))$ solution to problem (9). Also, the map $\phi \mapsto v$ is continuous from $H^s(\mathbb{R}^2)$ to $C([0,T],\ H^s(\mathbb{R}^2)$. Now, from the properties of group $Q(-t)$ it can be verified that $u(t) = Q(-t)v(t)$ is solution to (8) and satisfies the properties enunciated in Theorem 3. □

Theorem 7. *The time of existence of the solution to* (8) *can be chosen independently from s in the following sense: if* $u \in C([0,T], H^s)$ *is the solution to* (8) *with* $u(0) = \phi \in H^r$, *for some* $r > s$, *then* $u \in C([0,T], H^r)$. *In particular, if* $\phi \in H^\infty$, $u \in C([0,T], H^\infty)$.

Proof. The proof of this result is essentially the same as part (c) of Theorem 1 in [9]. We will briefly outline this. Let $r > s$, $u \in C([0,T], H^s)$ be the solution to (8) and $v = Q(t)u$. Let us suppose that $r \leq s+1$. Applying ∂_x^2 in both sides of the differential equation in (9), we arrive at the following linear evolution equation in $w(t) = \partial_x^2 v(t)$,

$$\frac{dw}{dt} + A(t)w + B(t)w = f(t), \tag{14}$$

where

$$A(t) = \partial_x Q(t)(u(t))^p Q(-t) \tag{15}$$

$$B(t) = 2Q(t)[p(u(t))^{p-1}]u_x(t)Q(-t) \tag{16}$$

$$f(t) = -Q(t)[p(p-1)u^{p-2}(t)][u_x(t)]^3. \tag{17}$$

Since $v \in C([0,T), H^s)$ we have that $w \in C([0,T); H^{s-2})$. Also, $w(0) = \phi_{xx} \in H^{r-2}$, because $\phi \in H^r$. Let us prove that $w \in C([0,T], H^{r-2})$. To do this, we shall prove that the Cauchy problem associated to the linear equation lineal (14) is well-posed for $1-s \leq k \leq s-1$. In this direction, we have the following lemma whose proof is completely similar to that of Lemma 3.1 in [9]. □

Lemma 8. *The family $\{A(t)\}_{0\leq t\leq T}$ has an unique family of evolution operators $U(t,\tau)_{0\leq t\leq \tau\leq T}$ in the spaces $X = H^h$, $Y = H^k$ (in the Kato sense), where*

$$-s \leq h \leq s-2 \quad 1-s \leq k \leq s-1 \quad k+1 \leq h. \tag{18}$$

In particular, $U(t,\tau) : H^r \to H^r$ for $-s \leq r \leq s-1$.

Then, the last lemma allows us to show that w satisfies the equation

$$w(t) = U(t,0)\phi_{xx} + \int_0^t U(t,\tau)[-B(\tau)w(\tau) + f(\tau)]d\tau. \tag{19}$$

Now, since $w(0) = \phi_{xx} \in H^{r-2}$, by (17), f is in $C([0,T], H^{s-1}) \subset C([0,T], H^{r-2})$ (if $r \leq s+1$) and $B(t)$, given in (16), is a family of operators in $\mathcal{B}(H^{r-2})$ strongly continuous for t in the interval $[0,T]$ (if $r \leq s+1$), from Lemma 8, the solution to (19) is in $C([0,T], H^{r-2})$ ((19) is an integral equation of Volterra type in H^{r-2}, which can be solved by successive approximations), in other words, $\partial_x^2 u \in C([0,T], H^{r-2})$.

If $w_1(t) = \partial_x\partial_y v(t)$, we have

$$\frac{dw_1}{dt} + A(t)w_1 + B_1(t)w_1 = f_1(t), \tag{20}$$

where

$$B_1(t) = Q(t)[p(u(t))^{p-1}]u_x(t)Q(-t) = \frac{1}{2}B(t) \tag{21}$$

$$f_1(t) = -Q(t)((p(p-1)u^{p-2}(t)[u_x(t)]^2 + p(u(t))^{p-1}u_{xx}(t))u_y(t)). \tag{22}$$

As above, we have

$$w_1(t) = U(t,0)\phi_{xy} + \int_0^t U(t,\tau)[-B_1(\tau)w_1(\tau) + f_1(\tau)]d\tau. \tag{23}$$

Inasmuch as $u_{xx} \in C([0,T], H^{r-2})$, $f_1 \in C([0,T], H^{r-2})$. Since, also, $B_1(t) \in \mathcal{B}(H^{r-2})$ is strongly continuous in the interval $[0,T]$, arguing as before, we have that $w_1 \in C([0,T], H^{r-2})$ or, equivalently, $u_{xy} \in C([0,T], H^{r-2})$.

Analogously, if $w_2(t) = \partial_y^2 v(t)$, we have

$$\frac{dw_2}{dt} + A(t)w_2 = f_2(t), \tag{24}$$

where

$$f_2(t) = -Q(t)((p(p-1)u^{p-2}(t)u_x(t)u_y(t) + 2p(u(t))^{p-1}u_{xy}(t))u_y(t)). \tag{25}$$

Then,

$$w_2(t) = U(t,0)\phi_{yy} + \int_0^t U(t,\tau)f_2(\tau)d\tau. \tag{26}$$

Since $u_{xy} \in C([0,T], H^{r-2})$, $f_2 \in C([0,T], H^{r-2})$. Repeating the argument above, we can conclude that $w_1 \in C([0,T], H^{r-2})$ or, equivalently, $\partial_y^2 u \in C([0,T], H^{r-2})$.

Then, we have proved that, if $s < r \leq s+1$ and $\phi \in H^r$, $u \in C([0,T], H^r)$. To the case $r > s+1$, as $\phi \in H^{s'}$, for $s' < r$, using a bootstrapping argument can be shown that $u \in C([0,T], H^r)$.

4. Remarks on the Persitence

The following result deals with the persistence of the solutions of (1) in X^s.

Theorem 9. *Let $s > 2$, p odd. And $u \in C([0,T]; X^s)$ is the solution of (1), corresponding to an initial data $\phi \in X^s$, then $u \equiv 0$.*

Proof. In view of the fact that $u \in C([0,T];X^s)$, $\hat{u}$ is continuous in η. Multiplying by y in (1) we obtain

$$\partial_t(yu) + y\mathcal{H}^{(y)}\partial_x^2 u + yu^p u_x = 0, \tag{27}$$

and since

$$\|yu^p u_x\|_0 \le |u^{p-1}u_x|_\infty \|yu\|_0 \le c\|u\|_s^p \|yu\|_0,$$

thus $yu^p u_x \in L^2$.

In the similar way, it can be shown that $yu^p u_x \in C([0,T];L^2)$.

Let us apply the Fourier transform to (27):

$$\begin{aligned} i\partial_t(\partial_\eta \widehat{u}) &= i\partial_\eta[\text{sgn}(\eta)\xi^2\widehat{u}] - \xi\partial_\eta \frac{\widehat{u^{p+1}}}{p+1} \\ &= i\delta(\eta)\xi^2\widehat{u} + i\text{sgn}(\eta)\xi^2\partial_\eta\widehat{u} - \xi\partial_\eta\frac{\widehat{u^{p+1}}}{p+1}. \end{aligned}$$

Now integrating with respect to t, we achieve that for every $t \in (0,T]$, the function

$$(\xi,\eta) \to \delta(\eta)\xi^2 \int_0^t \widehat{u}(\tau,\xi,\eta)d\tau$$

is measurable. Hence we must have

$$\int_0^t \widehat{u}(\tau,\cdot,0)d\tau = 0, \quad \forall t \in [0,T], \tag{28}$$

and therefore $\widehat{u}(t,\cdot,0) = 0$, $\forall t \in [0,T]$.

Taking into account the boundedness of the Fourier transform we obtain, for $t \in [0,T]$, $\eta \in \mathbb{R}$ and *a.e.* $\xi \in \mathbb{R}$:

$$\begin{aligned} \widehat{u}(t,\xi,\eta) &= \widehat{u}(0,\xi,\eta) - \int_0^t \widehat{\overbrace{\mathcal{H}^{(y)}\partial_x^2 u + u^p u_x}}(\tau,\xi,\eta)d\tau \\ &= \widehat{u}(0,\xi,\eta) + \int_0^t [i\text{sgn}(\eta)(\xi^2)\widehat{u} + i\frac{\xi}{p+1}\widehat{u^{p+1}}](\tau,\xi,\eta)d\tau. \end{aligned}$$

Evaluating at $\eta = 0$, one gets

$$\int_0^t \widehat{u^{p+1}}(\tau,\xi,0)d\tau = 0, \quad \forall t \in [0,T], \quad \xi - a.e.$$

It is clear that

$$\int u^{p+1}(t,x,y)dy = 0, \forall x \in \mathbb{R}, \quad \forall t \in [0,T].$$

And the result follows at once. □

Corollary 10. *Let $u \in C([0.T]; H^s(\mathbb{R}^2))$, $s > 2$, be a solution of* (1), *and p odd. If there exists $R > 0$ such that*

$$\operatorname{supp} u(t,x,\cdot) \subset (-R,R), \quad \forall x \in \mathbb{R},$$

and for each $t \in [t_1,t_2]$, $t_1 < t_2$, $t_1,t_2 \in [0,T]$, then $u \equiv 0$ in the interval $[t_1,t_2]$.

A proof of this corollary can be seen in [10].

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