A NEW RANDOM APPROACH TO
THE LEBESGUE INTEGRAL

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Abstract: In this paper a new method is given for generating the Lebesgue
integral. For the corresponding random Riemann sums it is shown that they
converge to the Lebesgue integral in probability.

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1. Introduction

The concept of random Riemann sums is introduced in [2] and [3] in the fol-
lowing manner.

Denote the interval [0, 1) by I and let I be equipped by Borel σ-algebra.
Let m be the Lebesgue measure on I. By a partition P₀ of I we mean a finite
sequence, x₀, x₁, ..., xₙ of elements of I such that 0 = x₀ < x₁ < ... < xₙ = 1.
The norm of P₀ with respect to the arbitrary measure µ on I is | P₀ |₁ :=
max{µ(I_k) : I_k = [x_{k-1}, x_k), 1 ≤ k ≤ n}.

For each I_k ∈ P₀, let tₖ ∈ I_k, 1 ≤ k ≤ n, be a random variable with uniform
distribution in the interval (x_{k-1}, x_k), t_k’s being independent.

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Let $f : I \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. The random Riemann sum of $f$ on $\mathcal{P}_0$ is

$$S_{\mathcal{P}_0}(f) = \sum f(t_k)m(I_k).$$

In [3] some results are proved for Lebesgue measure $m$. As an example, Proposition 2.1. of [3], can be mentioned which is equivalent to the following

**Theorem 1.** For any $\epsilon > 0$, and any sequence of partitions $\mathcal{P}_n$, $n \geq 1$, if

$$\lim_{n \to \infty} |\mathcal{P}_n|_m = 0,$$

then

$$P(\left| S_{\mathcal{P}_n}(f) - \int_I f dm \right| > \epsilon) \longrightarrow 0.$$
and hence if we take
\[ g = f(T_1, X_1, T_2, \ldots, X_n, T_{n+1}) (t_1, x_1, t_2, \ldots, x_n, t_{n+1}) \]
and \( h = f(X_1, \ldots, X_n) (x_1, \ldots, x_n) \), then we have
\[ g = h f_{T_1|X_1=x_1}(t_1) f_{T_2|X_2=x_2, X_3=x_3}(t_2) \ldots f_{T_n|X_n=x_n}(t_n), \]
i.e.
\[ g = \begin{cases} \frac{n!}{x_2-x_1 \ldots (x_n-x_{n-1}) (1-x_n)}, & \text{if } 0 \leq t_1 < t_2 < \ldots < x_n \leq t_{n+1} < 1; \\ 0, & \text{if otherwise.} \end{cases} \]

Now let \((\Omega_0, \mathcal{B}_0, P_0)\) be s.t. \(\Omega_0\) is the set of elements of \(I^{2n+1}\) with distinct coordinates and \(\mathcal{B}_0\) the Borel \(\sigma\)-algebra in it. Define for \(A \in \mathcal{B}_0\),
\[ P_0(A) = \int_{A \cap \{0<T_1<X_1<T_2<\ldots<X_n<T_{n+1}<1\}} g dt_1 dx_1 dt_2 \ldots dx_n dt_{n+1}. \]

**Lemma 2.** Let \(k_1, k_2, \ldots\) be an increasing sequence of natural numbers. There is a probability space \((\Omega, \mathcal{B}, P)\) for which each realization of an outcome yields a sequence \(\{A_n\}_{n \geq 1}\), where \(A_n, n \geq 1\), is a strictly increasing sequence like \(0 = x_0^{(n)}, t_1^{(n)}, x_1^{(n)}, t_2^{(n)}, \ldots, x_k^{(n)}, t_{k+1}^{(n)}, x_{k+1}^{(n)} = 1\) s.t. for each \(n\), \(x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}\) are the corresponding ordered statistics of a random sample from uniform distribution in \(I\) and for each \(n\), given \((x_1^{(n)}, x_2^{(n)}, \ldots, x_{k+1}^{(n)}), t_i^{(n)}\) \(s\) are independent each having uniform distribution in \([x_{i-1}^{(n)}, x_i^{(n)}]\). Moreover \(A_n s, n \geq 1\), form a mutually independent sequence of random vectors.

**Proof.** According to the previous lemma for each \(n \geq 1\), there is a probability space \((\Omega_n, \mathcal{B}_n, P_n)\) which yields the random vector
\[ (t_1^{(n)}, x_1^{(n)}, t_2^{(n)}, \ldots, x_{k+1}^{(n)}, t_{k+1}^{(n)}). \]
Now let \((\Omega, \mathcal{B}, P)\) be s.t. \(\Omega = \Omega_1 \Omega_2 \ldots \Omega_n \ldots\) and \(\mathcal{B}\) the Borel \(\sigma\)-algebra in it and take \(P = P_1 \otimes P_2 \otimes \ldots \otimes P_n \otimes \ldots\)

**Lemma 3.** For the sequence of partitions \(\{P_n\}_{n \geq 1}\), if \(P_n\) is constituted of points \(0, X_1^{(n)}, X_2^{(n)}, \ldots, X_{k+1}^{(n)}, 1\), where \(X_i^{(n)}\)'s and and \(k_i s\) being as described in the above lemma, then \(|P_n|\) tends to zero with probability 1.
Proof. It is sufficient to prove that, a.s., $\bigcup_{n \geq 1} P_n$ is dense in $I$. For fixed arbitrary sub-interval $(a, b)$ of $I$, let $G_n$ be the event of the sequence $\{P_i\}_{i \geq 1}$, having at least one point in $(a, b)$, in the $n$-th term for the first time. The assertion will be proved if we show that $P(\bigcup_{n \geq 1} G_n) = 1$. We have

$$P(\bigcup_{n \geq 1} G_n) = \sum_{n \geq 1} P(G_n) = \sum_{n \geq 1} (1 - (b - a))^{k_1 + k_2 + \ldots + k_n - 1} (1 - (1 - (b - a))^{k_n}).$$

It is clear that the above series tends to one. So $\bigcup_{n \geq 1} P_n$ is dense in $I$ and the truth of result is obvious.

For partition $\{P_n\}_{n \geq 1}$, define

$$Y_k = f(t_k^{(n)}(x_k^{(n)} - x_{k-1}^{(n)})) | (X_k^{(n)} = x_k, X_{k-1}^{(n)} = x_{k-1}), k \geq 1,$$

and $SP_n(f) = \sum Y_k$. We have

$$E(f(t_k^{(n)}(x_k^{(n)} - x_{k-1}^{(n)})) | (X_k^{(n)} = x_k, X_{k-1}^{(n)} = x_{k-1})) = E(f(t_k^{(n)}(x_k^{(n)} - x_{k-1}^{(n)}))) = \int_{I_k} f dm,$$

and so

$$E(SP_n(f)) = \sum \int_{I_k} f dm = \int_I f dm.$$

The main theorem is the following, based on Lemmas 1,2,3 in this paper and Proposition 2.1. in [3] which is coming, in the sequel.

**Theorem 2.** Suppose $f : I \rightarrow \mathbb{R}$ is a Lebesgue integrable function. For the sequence of random partitions $\{P_n\}_{n \geq 1}$, if $|P_n|$ tends to zero with probability 1, when $n \rightarrow \infty$, we have

$$P(|SP_n(f) - \int_I f dm| > \epsilon) < \epsilon, \text{ for all } \epsilon > 0.$$

**Remark 1.** In the same manner, after the same conditions other results in [3] can be seen to hold naturally.

**References**

