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SUM AND DIFFERENCE OF INDEPENDENT BETA TYPE 3 VARIABLES

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Abstract: In this article, we derive density functions of the sum and the difference of two independent random variables each distributed as beta type 3. We express these densities in terms of Appell's first hypergeometric function F_1.

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1. Introduction

The beta type 1 distribution with parameters (a, b) is defined by the probability density function (p.d.f.)

$$\mathrm{B1}(u; a, b) = \{B(a, b)\}^{-1} u^{a-1}(1-u)^{b-1}, \quad 0 < u < 1, \tag{1}$$

where $a > 0$, $b > 0$, and $B(a, b)$ is the beta function defined by

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$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0.$$

The beta type 1 distribution is well known in Bayesian methodology as a prior distribution on the success probability of a binomial distribution. The random variable V with the p.d.f.

$$\mathrm{B2}(v;a,b) = \{B(a,b)\}^{-1} v^{a-1}(1+v)^{-(a+b)}, \quad v > 0, \tag{2}$$

where $a > 0$ and $b > 0$, is said to have a beta type 2 distribution with parameters (a,b). Since (2) can be obtained from (1) by the transformation $V = U/(1-U)$ some authors call the distribution of V an *inverted beta distribution*. The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, e.g., see Johnson, Kotz and Balakrishnan [6]. Systematic treatment of matrix variate generalizations of beta distributions is given in Gupta and Nagar [3]. By using the transformation $W = U/(2-U)$, the beta type 3 density is obtained as (Gupta and Nagar [4, 5], Cardeño, Nagar and Sánchez [2]),

$$\mathrm{B3}(w;a,b) = 2^a\{B(a,b)\}^{-1} w^{a-1}(1-w)^{b-1}(1+w)^{-(a+b)}, \quad 0 < w < 1, \tag{3}$$

where $a > 0$ and $b > 0$.

It is well known that if X and Y are independent random variables having a standard gamma distribution with shape parameters a and b, respectively, then $X/(X+Y) \sim \mathrm{B1}(a,b)$, $X/Y \sim \mathrm{B2}(a,b)$ and $X/(X+2Y) \sim \mathrm{B3}(a,b)$.

In this article, we give distributions of sum and difference of two independent radon variables each having beta type 3 distribution. The distributions of the product and the ratio of two independent random variables when at least one of them is beta type 3 are available in Sánchez and Nagar [9].

2. Some Known Definitions and Results

The first hypergeometric function of Appell is defined in a series form as

$$F_1(a,b_1,b_2;c;z_1,z_2) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \frac{(a)_{r_1+r_2}(b_1)_{r_1}(b_2)_{r_1}}{(c)_{r_1+r_2}} \frac{z_1^{r_1} z_2^{r_2}}{r_1!\, r_2!}, \tag{4}$$

where $|z_1| < 1$, $|z_2| < 1$ and the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, with $(a)_0 = 1$.

From (4), it follows that

$$F_1(a,b_1,b_2;c;z_1,z_2)=\sum_{r_1=0}^{\infty}\frac{(a)_{r_1}(b_1)_{r_1}}{(c)_{r_1}}\frac{z_1^{r_1}}{r_1!}\ {}_2F_1(a+r_1,b_2;c+r_1;z_2)$$
$$=\sum_{r_2=0}^{\infty}\frac{(a)_{r_2}(b_2)_{r_2}}{(c)_{r_2}}\frac{z_2^{r_2}}{r_2!}\ {}_2F_1(a+r_2,b_1;c+r_2;z_1),$$

where ${}_2F_1$ is the Gauss hypergeometric function. The integral representation of the Gauss hypergeometric function is given as (Luke [7, Eq. 3.6(1)]),

$${}_2F_1(a,b;c;z)=\frac{1}{B(a,c-a)}\int_0^1\frac{t^{a-1}(1-t)^{c-a-1}}{(1-zt)^b}\,\mathrm{d}t, \tag{5}$$

where $\mathrm{Re}(c)>\mathrm{Re}(a)>0$ and $|\arg(1-z)|<\pi$. Expanding $(1-zt)^{-b}$, $|zt|<1$, in (5) and integrating t, the series expansion for ${}_2F_1$ is derived as

$${}_2F_1(a,b;c;z)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!}. \tag{6}$$

Further, writing

$$\frac{(a)_{r_1+r_2}}{(c)_{r_1+r_2}}=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 v^{a+r_1+r_2-1}(1-v)^{c-a-1}\,\mathrm{d}v,\quad \mathrm{Re}(c)>\mathrm{Re}(a)>0$$

and

$$\sum_{r_i=0}^{\infty}\frac{(b_i)_{r_i}(vz_i)^{r_i}}{r_i!}=(1-vz_i)^{-b_i},\quad |vz_i|<1,\quad i=1,2$$

in (4), one can derive an integral representation of the first hypergeometric function of Appell as

$$F_1(a,b_1,b_2;c;z_1,z_2)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{v^{a-1}(1-v)^{c-a-1}\,\mathrm{d}v}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}},$$
$$|z_1|<1,\quad |z_2|<1,\quad \mathrm{Re}(c)>\mathrm{Re}(a)>0. \tag{7}$$

For further results the reader is referred to Bailey [1] and Srivastava and Karlsson [10].

To derive our distributional results we need the following theorem given in Nadarajah [8].

Theorem 2.1. *Let X_1 and X_2 be independent random variables having support in $(0,1)$ and densities f_1 and f_2, respectively. Then, the densities of $S = X_1 + X_2$ and $D = X_1 - X_2$ can be expressed as*

$$f_S(s) = \begin{cases} \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t, & \text{if} \quad 0 < s < 1, \\ \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t, & \text{if} \quad 1 < s < 2, \end{cases}$$

and

$$f_D(d) = \begin{cases} \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t, & \text{if} \quad -1 < d < 0, \\ \int_0^{1-d} f_1(d+t) f_2(t)\,\mathrm{d}t, & \text{if} \quad 0 < d < 1, \end{cases}$$

respectively.

3. Densities of Sum and Difference

In this section, we give densities of the sum and difference of two independent random variables each having beta type 3 distribution.

Theorem 3.1. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{B3}(a_i, b_i)$, $i = 1, 2$. Then, the p.d.f. of $S = X_1 + X_2$ is derived as*

$$\begin{aligned} f_S(s) = {} & \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{s^{a_1+a_2-1}}{(1+s)^{a_2+b_2}} \\ & \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-b_2+1)_n}{m!\,n!} s^{m+n} \frac{\Gamma(a_1+m)\Gamma(a_2+n)}{\Gamma(a_1+a_2+m+n)} \\ & \times F_1\left(a_1+m, a_1+b_1, a_2+b_2; a_1+a_2+m+n; -s, \frac{s}{1+s}\right), \end{aligned}$$

for $0 < s < 1$, and

$$\begin{aligned} f_S(s) = {} & \frac{2^{a_1-b_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(2-s)^{b_1+b_2-1}}{s^{a_1+b_1}} \\ & \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-a_1+1)_m(-a_2+1)_n}{m!\,n!} (2-s)^{m+n} \frac{\Gamma(b_1+m)\Gamma(b_2+n)}{\Gamma(b_1+b_2+m+n)} \\ & \times F_1\left(b_2+n, a_1+b_1, a_2+b_2; b_1+b_2+m+n; -\frac{2-s}{s}, \frac{2-s}{2}\right), \end{aligned}$$

for $1 < s < 2$.

Proof. From Theorem 2.1, the density of S is given by

$$f_S(s) = \begin{cases} \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t, & \text{for} \quad 0 < s < 1, \\ \\ \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t, & \text{for} \quad 1 < s < 2, \end{cases}$$

and therefore in our derivation we consider cases $0 < s < 1$ and $1 < s < 2$ separately. Using (3), the density of S, for $0 < s < 1$, is derived as

$$\begin{aligned} f_S(s) &= \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t \\ &= s \int_0^1 f_1(sw) f_2(s(1-w))\,\mathrm{d}w \\ &= \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{s^{a_1+a_2-1}}{(1+s)^{a_2+b_2}} \\ &\quad \times \int_0^1 \frac{w^{a_1-1}(1-w)^{a_2-1}(1-sw)^{b_1-1}[1-s(1-w)]^{b_2-1}}{(1+sw)^{a_1+b_1}\,[1-sw/(1+s)]^{a_2+b_2}}\,\mathrm{d}w. \end{aligned} \tag{8}$$

Now, expanding $(1-sw)^{b_1-1}$ and $[1-s(1-w)]^{b_2-1}$ in the power series as

$$(1-sw)^{b_1-1}[1-s(1-w)]^{b_2-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-b_2+1)_n}{m!\,n!}(sw)^m\,[s(1-w)]^n$$

and applying (7), the integral in (8) is evaluated as

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-b_2+1)_n}{m!\,n!} s^{m+n} \int_0^1 \frac{w^{a_1+m-1}(1-w)^{a_2+n-1}}{(1+sw)^{a_1+b_1}\,[1-sw/(1+s)]^{a_2+b_2}}\,\mathrm{d}w \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-b_2+1)_n}{m!\,n!} s^{m+n} \frac{\Gamma(a_1+m)\Gamma(a_2+n)}{\Gamma(a_1+a_2+m+n)} \\ &\quad \times F_1\left(a_1+m, a_1+b_1, a_2+b_2; a_1+a_2+m+n, -s, \frac{s}{1+s}\right). \end{aligned} \tag{9}$$

Finally, substituting (9) in (8), we get the desired result. Similarly, the density of S, for $1 < s < 2$, is obtained as

$$\begin{aligned} f_S(s) &= \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t \\ &= (2-s) \int_0^1 f_1(s-1+(2-s)w) f_2(1-(2-s)w)\,\mathrm{d}w \end{aligned}$$

$$= \frac{2^{a_1-b_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(2-s)^{b_1+b_2-1}}{s^{a_1+b_1}}$$
$$\times \int_0^1 \frac{w^{b_2-1}(1-w)^{b_1-1}[1-(2-s)(1-w)]^{a_1-1}[1-(2-s)w]^{a_2-1}}{[1+(2-s)w/s]^{a_1+b_1}[1-(2-s)w/2]^{a_2+b_2}}\,\mathrm{d}w.$$

Replacing $[1-(2-s)(1-w)]^{a_1-1}$ and $[1-(2-s)w]^{a_2-1}$ by their respective power series expansions in the above integral and applying (7), one arrives at

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-a_1+1)_m(-a_2+1)_n}{m!\,n!}(2-s)^{m+n}$$
$$\times \int_0^1 \frac{w^{b_2+n-1}(1-w)^{b_1+m-1}}{[1+(2-s)w/s]^{a_1+b_1}[1-(2-s)w/2]^{a_2+b_2}}\,\mathrm{d}w$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-a_1+1)_m(-a_2+1)_n}{m!\,n!}(2-s)^{m+n} \frac{\Gamma(b_1+m)\Gamma(b_2+n)}{\Gamma(b_1+b_2+m+n)}$$
$$\times F_1\left(b_2+n, a_1+b_1, a_2+b_2; b_1+b_2+m+n; -\frac{2-s}{s}, \frac{2-s}{2}\right).$$

Finally, substituting appropriately, we get the desired result. □

Theorem 3.2. *Let X_1 and X_2 be independent random variables, $X_i \sim$ B3(a_i, b_i), $i = 1, 2$. Then, the p.d.f. of $D = X_1 - X_2$ is derived as*

$$f_D(d) = \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(1+d)^{a_1+b_2-1}}{(1-d)^{a_2+b_2}}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-a_2+1)_n}{m!\,n!}(1+d)^{m+n} \frac{\Gamma(a_1+m)\Gamma(b_2+n)}{\Gamma(a_1+b_2+m+n)}$$
$$\times F_1\left(a_1+m, a_1+b_1, a_2+b_2; a_1+b_2+m+n; -(1+d), -\frac{1+d}{1-d}\right)$$

for $-1 < d < 0$, and

$$f_D(d) = \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(1-d)^{a_2+b_1-1}}{(1+d)^{a_1+b_1}}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_2+1)_m(-a_1+1)_n}{m!\,n!}(1-d)^{m+n} \frac{\Gamma(a_2+m)\Gamma(b_1+n)}{\Gamma(a_2+b_1+m+n)}$$
$$\times F_1\left(a_2+m, a_1+b_1, a_2+b_2; a_2+b_1+m+n; -\frac{1-d}{1+d}, -(1-d)\right)$$

for $0 < d < 1$.

Proof. Theorem 2.1 states the density of D as

$$f_D(d) = \begin{cases} \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t, & \text{for} \quad -1 < d < 0, \\ \int_0^{1-d} f_1(d+t) f_2(t)\,\mathrm{d}t, & \text{for} \quad 0 < d < 1, \end{cases}$$

and therefore we consider cases $-1 < d < 0$ and $0 < d < 1$ separately. Using the beta type 3 density given in (3), the pdf of D, for $-1 < d < 0$, is derived as

$$\begin{aligned} f_D(d) &= \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t \\ &= (1+d)\int_0^1 f_1((1+d)w) f_2((1+d)w-d)\,\mathrm{d}w \\ &= \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(1+d)^{a_1+b_2-1}}{(1-d)^{a_2+b_2}} \\ &\quad \times \int_0^1 \frac{w^{a_1-1}(1-w)^{b_2-1}[1-(1+d)(1-w)]^{a_2-1}[1-(1+d)w]^{b_1-1}}{[1+(1+d)w]^{a_1+b_1}[1+(1+d)w/(1-d)]^{a_2+b_2}}\mathrm{d}w. \quad (10) \end{aligned}$$

Using the power series expansion

$$\begin{aligned} &[1-(1+d)w]^{b_1-1}[1-(1+d)(1-w)]^{a_2-1} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-a_2+1)_n}{m!\,n!}(1+d)^{m+n} w^m (1-w)^n \end{aligned}$$

and applying (7), the above integral is evaluated as

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-a_2+1)_n}{m!\,n!}(1+d)^{m+n} \\ &\times \int_0^1 \frac{w^{a_1+m-1}(1-w)^{b_2+n-1}}{[1+(1+d)w]^{a_1+b_1}[1+(1+d)w/(1-d)]^{a_2+b_2}}\mathrm{d}w \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-a_2+1)_n}{m!\,n!}(1+d)^{m+n} \frac{\Gamma(a_1+m)\Gamma(b_2+n)}{\Gamma(a_1+b_2+m+n)} \\ &\quad \times F_1\left(a_1+m, a_1+b_1, a_2+b_2; a_1+b_2+m+n; -(1+d), -\frac{1+d}{1-d}\right). \quad (11) \end{aligned}$$

Finally, substituting (11) in (10), we get the desired result. Using (3) and

Theorem 2.1, the density of D, for $0 < d < 1$, is derived as

$$\begin{aligned} f_D(d) &= \int_0^{1-d} f_1(d+t) f_2(t) \mathrm{d}t \\ &= (1-d) \int_0^1 f_1(d+(1-d)w) f_2((1-d)w)\, \mathrm{d}w \\ &= \frac{2^{a_1+a_2}}{B(a_1,b_1)B(a_2,b_2)} \frac{(1-d)^{a_2+b_1-1}}{(1+d)^{a_1+b_1}} \\ &\quad \times \int_0^1 \frac{w^{a_2-1}(1-w)^{b_1-1}[1-(1-d)(1-w)]^{a_1-1}[1-(1-d)w]^{b_2-1}}{[1+(1-d)w/(1+d)]^{a_1+b_1}[1+(1-d)w]^{a_2+b_2}} \mathrm{d}w. \quad (12) \end{aligned}$$

Now, expanding $[1-(1-d)w]^{b_2-1}[1-(1-d)(1-w)]^{a_1-1}$ in power series, the above integral is expressed as

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_2+1)_m(-a_1+1)_n}{m!\,n!}(1-d)^{m+n} \\ &\times \int_0^1 \frac{w^{a_2+m-1}(1-w)^{b_1+n-1}}{[1+(1-d)w/(1+d)]^{a_1+b_1}[1+(1-d)w]^{a_2+b_2}} \mathrm{d}w \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_2+1)_m(-a_1+1)_n}{m!\,n!}(1-d)^{m+n} \frac{\Gamma(a_2+m)\Gamma(b_1+n)}{\Gamma(a_2+b_1+m+n)} \\ &\quad \times F_1\left(a_2+m, a_1+b_1, a_2+b_2; a_2+b_1+m+n; -\frac{1-d}{1+d}, -(1-d)\right), \quad (13) \end{aligned}$$

where the last line has been obtained by using (7). Finally, substitution for the integral from (13) in (12) yields the desired result. □

Finally, in Theorem 3.3 and Theorem 3.4, we give densities of $X_1 + X_2$ and $X_1 - X_2$ where X_1 and X_2 are independent, $X_1 \sim \mathrm{B3}(a_1, b_1)$ and $X_2 \sim \mathrm{B1}(a_2, b_2)$. The proofs of these theorem are similar to those of Theorem 3.1 and Theorem 3.2 and can be constructed easily.

Theorem 3.3. *Let X_1 and X_2 be independent random variables, $X_1 \sim \mathrm{B3}(a_1, b_1)$ and $X_2 \sim \mathrm{B1}(a_2, b_2)$. Then, the p.d.f. of $S = X_1 + X_2$ is derived as*

$$\begin{aligned} &\frac{2^{a_1}}{B(a_1,b_1)B(a_2,b_2)} s^{a_1+a_2-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b_1+1)_m(-b_2+1)_n}{m!\,n!} s^{m+n} \\ &\quad \times B(a_1+m, a_2+n)\, {}_2F_1\left(a_1+m, a_1+b_1; a_1+a_2+m+n; -s\right), \end{aligned}$$

for $0 < s < 1$, *and*

$$\frac{2^{a_1}}{B(a_1,b_1)B(a_2,b_2)}\frac{(2-s)^{b_1+b_2-1}}{s^{a_1+b_1}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-a_1+1)_m(-a_2+1)_n}{m!\,n!}(2-s)^{m+n}$$
$$\times B(b_1+m,b_2+n)\,{}_2F_1\left(b_2+n,a_1+b_1;b_1+b_2+m+n;-\frac{2-s}{s}\right),$$

for $1 < s < 2$.

Theorem 3.4. *Let* X_1 *and* X_2 *be independent random variables,* $X_1 \sim \mathrm{B3}(a_1,b_1)$ *and* $X_2 \sim \mathrm{B1}(a_2,b_2)$. *Then, the p.d.f. of* $D = X_1 - X_2$ *is derived as*

$$\frac{2^{a_1}}{B(a_1,b_1)B(a_2,b_2)}(1+d)^{a_1+b_2-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-b_1+1)_m(-a_2+1)_n}{m!\,n!}(1+d)^{m+n}$$
$$\times B(a_1+m,b_2+n){}_2F_1\left(a_1+m,a_1+b_1;a_1+b_2+m+n;-(1+d)\right)$$

for $-1 < d < 0$, *and*

$$\frac{2^{a_1}}{B(a_1,b_1)B(a_2,b_2)}\frac{(1-d)^{a_2+b_1-1}}{(1+d)^{a_1+b_1}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-b_2+1)_m(-a_1+1)_n}{m!\,n!}(1-d)^{m+n}$$
$$\times B(a_2+m,b_1+n)\,{}_2F_1\left(a_2+m,a_1+b_1;a_2+b_1+m+n;-\frac{1-d}{1+d}\right)$$

for $0 < d < 1$.

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