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POSITIVELY HOMOGENEOUS FUNCTIONS AND MAXIMAL ABSTRACT MONOTONICITY

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Abstract: In this paper, by using a set of positively homogeneous functions, we develop a theory of monotone operators in the framework of abstract monotonicity. Indeed, by using a set of positively homogeneous functions, we give some criteria for maximal abstract monotonicity.

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1. Introduction

Positively homogeneous functions of degree one have found many applications in various areas of mathematics. In particular, positively homogeneous functions are used in nonsmooth analysis and nonsmooth optimization, since an approximation of nonsmooth functions can often be accomplished using positively homogeneous functions (see, for example, [6]). In this paper, by using a set of positively homogeneous functions, we investigate maximal abstract

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monotonicity. Several approaches to the theory of monotone operators have established links between maximal monotone operators and convex functions (see [1, 4, 6]). The richness of the theory of monotone operators has given rise to a great number of works and the simplification of proof and theory that has results from the use of convex analysis techniques justifies an interest in these links. Recently many authors have explored the use of convex representative functions in the study of monotone operators, e.g., [1, 4]. Roughly speaking the study of monotone operators is reduced to the study of the convexity of the coupling function, restricted to the monotone set.

The aim of the present paper is to develop a theory of monotone operators in the framework of abstract convexity by using a set of positively homogeneous functions.

The structure of the paper is as follows: In Section 2, we provide some preliminary definitions and results related to positively homogeneous functions and abstract monotonicity. In Section 3, by using a set of positively homogeneous functions, we give some criteria for maximal abstract monotonicity.

2. Preliminaries and Abstract Monotonicity

Let Y be a set and $L := \{l : Y \longrightarrow \mathbb{R} : l \text{ is a function}\}$ be a set of real-valued functions defined on Y. In the following, we present some definitions and properties of abstract monotone operators (see [5, 7]).

(i) A set valued mapping $T : Y \longrightarrow 2^L$ is called L-monotone operator (or, abstract monotone operator) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{2.1}$$

for all $l \in Tx$, $l' \in Tx'$ and all $x,\ x' \in Y$.

It is worth noting that if Y is a Banach space with the dual space Y^* and $L := Y^*$. Then, T is called monotone operator in the classical case.

(ii) A set valued mapping $T : Y \longrightarrow 2^L$ is called maximal L-monotone operator (or, maximal abstract monotone operator) if T is L-monotone and $T = T'$ for any L-monotone operator $T' : Y \longrightarrow 2^L$ such that $G(T) \subseteq G(T')$.

(iii) A subset S of $Y \times L$ is called L-monotone (or, abstract monotone) if $l(x) - l(x') - l'(x) + l'(x') \geq 0$ for all $(x, l),\ (x', l') \in S$.

(iv) A subset S of $Y \times L$ is called maximal L-monotone (or, maximal abstract monotone) if S is L-monotone and $S = S'$ for any L-monotone set S' such that $S \subseteq S'$.

Now, for a set Y, define $\tilde{L} := \{l : Y \longrightarrow \mathbb{R} : l \text{ is an abstract linear function}\}$ be a set of real valued abstract linear functions defined on Y. For each $l \in \tilde{L}$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of l on the constant $c : h_{l,c}(x) := l(x) - c, \quad (x \in Y)$. The function $h_{l,c}$ is called $\tilde{L}$-affine. Recall (see [6]) that the set $\tilde{L}$ is called a set of abstract linear functions if $h_{l,c} \notin \tilde{L}$ for all $l \in \tilde{L}$ and all $c \in \mathbb{R} \setminus \{0\}$. The set of all $\tilde{L}$-affine functions will be denoted by $H_{\tilde{L}}$. If $\tilde{L}$ is a set of abstract linear functions, then $h_{l,c} = h_{l_0,c_0}$ if and only if $l = l_0$ and $c = c_0$.

Let $f : Y \longrightarrow (-\infty, +\infty]$ be a function and $x_0 \in dom f$. Recall (see [6]) that an element $l \in \tilde{L}$ is called an $\tilde{L}$-subgradient of f at x_0 if $f(x) \geq f(x_0) + l(x) - l(x_0)$ for all $x \in Y$. The set $\partial_{\tilde{L}} f(x_0)$ of all $\tilde{L}$-subgradients of f at x_0 is called $\tilde{L}$-subdifferential of f at x_0. The subdifferential $\partial_{\tilde{L}} f(x_0)$ (see [[6], Proposition 1.2]) is non-empty if and only if $x_0 \in dom f$ and $f(x_0) = \max\{h(x_0) : h \in \text{supp } (f, H_{\tilde{L}})\}$. In the sequel, let Y be a conic set ($x \in Y,\ \lambda > 0 \implies \lambda x \in Y$). An extended real valued function $f : Y \longrightarrow [-\infty, +\infty]$ is called positively homogeneous (of degree one) if $f(\lambda x) = \lambda f(x)$ for all $x \in Y$ and all $\lambda > 0$. Assume that the conic set Y contains an element y such that $f(y) = 0$ for all positively homogeneous function $f : Y \longrightarrow (-\infty, +\infty]$, and for each $x \in Y$, $x \neq y$ there exists a positively homogeneous function $g : Y \longrightarrow (-\infty, +\infty]$ such that $g(x) \neq 0$. In such a case, the element y is called zero and is denoted by 0. We accept the following definition: $0.x = 0$ for all $x \in Y$.

Now, assume that the conic set Y containing zero. Let

$$\tilde{L} := \{l : Y \longrightarrow \mathbb{R} : l \text{ is a positively homogeneous function}\} \tag{2.2}$$

be a set of real valued positively homogeneous functions defined on Y. It follows from the positively homogeneity that $\tilde{L}$ is a set of abstract linear functions. In the following, we give an example of a maximal $\tilde{L}$-monotone operator. The proof of the following theorem is similar to the one of Theorem 3.2 in [3], and therefore we omit it.

Theorem 2.1. *Let $f : Y \to (-\infty, +\infty]$ be a positively homogeneous function. Then, $\partial_{\tilde{L}} f$ is a maximal $\tilde{L}$-monotone operator, where $\tilde{L}$ is defined by (2.2).*

Recall that a function $p : X \longrightarrow \mathbb{R}$ is called sublinear if

$$p(\lambda x) = \lambda p(x) \quad (\lambda \geq 0,\ x \in X), \qquad p(x + y) \leq p(x) + p(y) \quad (x,\ y \in X).$$

Also, a function $q : X \longrightarrow \mathbb{R}$ is called superlinear if

$$q(\lambda x) = \lambda q(x) \quad (\lambda \geq 0,\ x \in X), \qquad q(x + y) \geq q(x) + q(y) \quad (x,\ y \in X),$$

where X is a vector space.

In the following, we gather a result which will be used later.

Lemma 2.1. (see [7], Theorem 7.2) *Let A be a non-empty convex subset of a vector space, F be a Banach space, $f : A \longrightarrow \mathbb{R}$ be a convex function and $g : A \longrightarrow F$ be an affine function. Then the following assertions are equivalent:*
(1) $f(a) + \|g(a)\|^2 \geq 0$ *for all* $a \in A$.
(2) *There exists* $y^* \in F^*$ *such that*

$$f(a) - 2\langle g(a), y^* \rangle - \|y^*\|^2 \geq 0, \quad \forall\, a \in A.$$

3. Abstract Monotonicity with Respect to a Set of Positively Homogeneous Functions

In this section, we present criteria for maximal abstract monotonicity with respect to a set of positively homogeneous functions, and also we obtain some related results.

From now on, let X be a Banach space. Let $S \subset X$ be a closed convex and pointed cone. (The latter means that $S \cap (-S) = \{0\}$.) The cone S generates an order relation $\geq$ on X. By definition $x \geq y \iff x - y \in S$. Assume that S is solid, that is, the interior $\operatorname{int} S$ of S is non-empty. Let $\mathbf{1} \in \operatorname{int} S$. Using $\mathbf{1}$ we can define the following function:

$$p(x) = \inf\{\lambda \in \mathbb{R} : x \leq \lambda \mathbf{1}\}, \qquad (x \in X). \tag{3.1}$$

It is easy to check that p is finite. It follows from (3.1) that

$$x \leq p(x)\mathbf{1}, \qquad (x \in X). \tag{3.2}$$

It is easy to see (and well-known) that p is a sublinear function. We need the following definition (see [6] and references therein). A function $s : X \to \mathbb{R}$ is called topical if s is increasing: $x \geq y$ implies $s(x) \geq s(y)$ and $s(x + \lambda \mathbf{1}) = s(x) + \lambda$ for all $x \in X$ and $\lambda \in \mathbb{R}$.

It follows from the definition of p that p is topical. Consider the function

$$\|x\| := \max(p(x), p(-x)). \tag{3.3}$$

It is easy to show (and well-known) that $\|\cdot\|$ is a norm on X. In the sequel, we assume that the norm (3.3) coincides with the norm of the Banach space X.

In the sequel, let

$$PH(X) := \{l : X \longrightarrow \mathbb{R} : l \text{ is a positively homogeneous function}\}$$

be the set of all real valued positively homogeneous (PH) functions defined on X. It is clear that $PH(X)$ is a linear vector (lattice) space. Let

$$K := \{l \in PH(X) : l(x) \geq 0 \quad \forall\, x \in X\}.$$

It is easy to see that K is an algebraic closed convex and pointed cone in $PH(X)$. The cone K generates an order relation $\geq$ on $PH(X)$. By definition, $l \geq l' \iff l - l' \in K$. Define the function $\mathbf{u} : X \longrightarrow \mathbb{R}$ by $\mathbf{u}(x) := \|x\|$ for all $x \in X$. It is clear that $\mathbf{u}$ is a positively homogeneous function on X, and hence $\mathbf{u} \in PH(X)$. Indeed, $\mathbf{u} \in K$ and $\mathbf{u}(\mathbf{1}) = \|\mathbf{1}\| = 1$ (see (3.3)), where $\mathbf{1} \in \operatorname{int} S$.

Lemma 3.1. *Let K^i be the algebraic interior of K. Then $\mathbf{u} \in K^i$.*

Proof. We must show that for each $l \in PH(X)$ there exists $\varepsilon > 0$ such that $\mathbf{u} + \varepsilon l \in K$. To do this, let $l \in PH(X)$ be arbitrary. Since $\mathbf{u} + \varepsilon l$ is positively homogeneous, it is enough to show that $\mathbf{u} + \varepsilon l \geq 0$ on S_X. If $l \geq 0$ on S_X, then it is clear that $\mathbf{u} + \varepsilon l \geq 0$ on S_X for each $\varepsilon > 0$. Assume that $\inf_{x \in S_X} l(x) < 0$. Let $A := \{x \in S_X : l(x) \geq 0\}$ and $B := \{x \in S_X : l(x) < 0\}$. Suppose that $0 < \varepsilon < -(\inf_{x \in S_X} l(x))^{-1}$. Then we have

$$\begin{aligned}
&\mathbf{u}(x) + \varepsilon l(x) \\
&\geq \inf_{x \in B} [\|x\| + \varepsilon l(x)] \\
&\geq \inf_{x \in S_X} [\|x\| + \varepsilon l(x)] \\
&\geq 1 + \varepsilon \inf_{x \in S_X} l(x) \\
&> 0, \quad \forall\, x \in B.
\end{aligned}$$

Hence, $\mathbf{u} + \varepsilon l \geq 0$ on S_X, and the proof is complete. □

Now, using $\mathbf{u}$ we can define the function: $q : PH(X) \longrightarrow \mathbb{R}$ by:

$$q(l) = \inf\{\lambda \in \mathbb{R} : l \leq \lambda \mathbf{u}\}, \qquad (l \in PH(X)). \tag{3.4}$$

It is easy to check that q is finite and has all properties of the function p.

Lemma 3.2. *We have*

$$-q(-l)\|x\| \leq l(x) \leq q(l)\|x\|, \quad \forall\, x \in X;\ \forall\, l \in PH(X), \tag{3.5}$$

and hence:

$$-\|l\|\|x\| \leq l(x) \leq \|l\|\|x\|, \quad \forall\, x \in X;\ \forall\, l \in PH(X). \tag{3.6}$$

Proof. It is obvious. □

Remark 3.1. Note that $|q(-l)| \leq \|l\|$ for all $l \in PH(X)$. Indeed, we have $-l \leq q(-l)\mathbf{u}$, and so $l \geq -q(-l)\mathbf{u}$. Since q is a topical function, it follows that $q(l) \geq -q(-l)$. Therefore, one has

$$\|l\| = \max(q(l), q(-l)) \geq \max(-q(-l), q(-l)) = |q(-l)|.$$

In the sequel, let L be any closed subspace of $PH(X)$ such that $X^* \subseteq L$ (X^* is the dual space of X) and each $l \in L$ is a sublinear or superlinear function.

Example 3.1. *Let $X := \mathbb{R}^n$ and $\mathbf{u} : X \longrightarrow \mathbb{R}$ be defined by $\mathbf{u}(x) = \|x\|$ for all $x \in X$. Let $a \in \mathbb{R}$ and $y \in X$ be arbitrary. Define the function $l_{y,a} : X \longrightarrow \mathbb{R}$ by*

$$l_{y,a}(x) := a\mathbf{u}(x) + [y, x], \quad \forall\, x \in X.$$

The function $l_{y,a}$ ($a \in \mathbb{R},\ y \in X$) has the following properties:
(1) *For each $a \in \mathbb{R}$ and $y \in X$, $l_{y,a}$ is a positively homogeneous function.*
(2) *$l_{y_1,a_1} + l_{y_2,a_2} = l_{y_1+y_2,a_1+a_2}$ for all $a_1,\ a_2 \in \mathbb{R}$ and all $y_1,\ y_2 \in X$.*
(3) *$\alpha l_{y,a} = l_{\alpha y,\alpha a}$ for all $a,\ \alpha \in \mathbb{R}$ and all $y \in X$.*
(4) *If $a \in \mathbb{R}$ and $a \geq 0$, then $l_{y,a}$ is a continuous sublinear function. If $a \in \mathbb{R}$ and $a < 0$, then $l_{y,a}$ is a continuous superlinear function.*

Now, let

$$L := \{l_{y,a} : a \in \mathbb{R},\ y \in X\}.$$

It is easy to see that L is a closed subspace of $PH(X)$ and $X^* \subseteq L$. Moreover, each $l_{y,a} \in L$ is a sublinear or superlinear function.

Define the coupling function $\langle ., . \rangle : X \times L \longrightarrow \mathbb{R}$ by

$$\langle x, l \rangle := l(x), \quad \forall\, x \in X;\ \forall\, l \in L.$$

For each $x \in X$, define the function $\gamma_x : L \longrightarrow \mathbb{R}$ by $\gamma_x(l) := l(x)$ for all $l \in L$. It is easy to see that γ_x is a continuous linear functional on L (see Remark 3.2, below). Let $\hat{L} := \{\gamma_x : x \in X\}$. Then we have $\hat{L} \subseteq L^*$, where L^* is the dual space of L.

Remark 3.2. Note that $\|\gamma_x\| \leq \|x\|$ for each $x \in X$.

Assumption $(\mathcal{R})$: *We assume that* $\hat{L} = L^*$, *where* L^* *is the dual space of* L.

Remark 3.3. Note that if $L := X^*$ (the dual space of X), then $\hat{L}$ is a Banach space and by Remark 3.2 and Hahn-Banach theorem we conclude that $\hat{L} = X$. Therefore, if X is reflexive, one has $\hat{L} = L^*$, that is Assumption $(\mathcal{R})$ holds.

We denote (see [7]) by $\mathbb{R}^{(X\times L)}$ the direct sum of $X \times L$ copies of $\mathbb{R}$, that is, the set of all functions $\alpha : X \times L \longrightarrow \mathbb{R}$ such that the set

$$\{(x, l) \in X \times L : \alpha(x, l) \neq 0\}$$

is finite. It is clear that $\mathbb{R}^{(X\times L)}$ is a vector space. For each $(x_0, l_0) \in X \times L$, define the function $\delta_{(x_0,l_0)} : X \times L \longrightarrow \mathbb{R}$ by:

$$\delta_{(x_0,l_0)}(x, l) := \begin{cases} 1, & \text{if} \quad (x, l) = (x_0, l_0) \\ 0, & \text{otherwise,} \end{cases}$$

for all $(x, l) \in X \times L$. Then, $\delta_{(x_0,l_0)} \in \mathbb{R}^{(X\times L)}$ for all $(x_0, l_0) \in X \times L$.
For a non-empty subset M of $X \times L$, we denote by $\mathrm{co}(M)$ the convex hull in $\mathbb{R}^{(X\times L)}$ of the set

$$\{\delta_{(x_0,l_0)} : (x_0, l_0) \in X \times L\}.$$

Explicitly, for $\alpha \in \mathbb{R}^{(X\times L)}$, we have $\alpha \in \mathrm{co}(M)$ if and only if

$$\alpha \geq 0 \text{ on } X \times L,\ \alpha(x, l) > 0 \implies (x, l) \in M \text{ and } \Sigma_{(x,l)\in M}\alpha(x, l) = 1.$$

In the following, define the functions $n : \mathbb{R}^{(X\times L)} \longrightarrow X$, $m : \mathbb{R}^{(X\times L)} \longrightarrow L$, and $r : \mathbb{R}^{(X\times L)} \longrightarrow \mathbb{R}$ by

$$n(\alpha) := \Sigma_{(x,l)\in X\times L}\alpha(x, l)x, \quad \alpha \in \mathbb{R}^{(X\times L)},$$

$$m(\alpha) := \Sigma_{(x,l)\in X\times L}\alpha(x, l)l, \quad \alpha \in \mathbb{R}^{(X\times L)},$$

and

$$r(\alpha) := \Sigma_{(x,l)\in X\times L}\alpha(x, l)l(x), \quad \alpha \in \mathbb{R}^{(X\times L)}.$$

It is clear that n, m and r are linear operators. Also, for each $(x,l) \in X \times L$, we have

$$n(\delta_{(x,l)}) = x, \quad m(\delta_{(x,l)}) = l, \quad \text{and} \quad r(\delta_{(x,l)}) = l(x). \tag{3.7}$$

Lemma 3.3. *Let $M \subseteq X \times L$ be a non-empty L-monotone set such that for each $(x,l) \in M$, we have l is a sublinear function. Then*

$$r(\alpha) \geq m(\alpha)(n(\alpha)) \quad \textit{for all } \alpha \in \mathrm{co}(M) \textit{ such that } m(\alpha) \textit{ is sublinear}, \tag{3.8}$$

and

$$r(\alpha) \geq -m(\alpha)(-n(\alpha)) \quad \textit{for all } \alpha \in \mathrm{co}(M) \textit{ such that } m(\alpha) \textit{ is superlinear}. \tag{3.9}$$

Proof. Let $\alpha \in \mathrm{co}(M)$ be fixed and arbitrary. Let $(x_1,l_1), \cdots, (x_k,l_k)$ be an enumeration of those elements $(x,l) \in M$ for which $\alpha(x,l) > 0$. Assume that $\alpha_i := \alpha(x_i,l_i) > 0,\ i = 1, \cdots, k$. Note that since $\alpha \in \mathrm{co}(M)$, it follows that $\Sigma_{i=1}^k \alpha_i = 1$. Since $m(\alpha) \in L$, one has $m(\alpha)$ is a sublinear or superliner function.

Case (i). Assume that $m(\alpha) \in L$ is a sublinear function. Thus, we have

$$\begin{aligned} & m(\alpha)(n(\alpha)) \\ &= m(\alpha)(\Sigma_{i=1}^k \alpha_i x_i) \\ &\leq \Sigma_{i=1}^k \alpha_i m(\alpha)(x_i) \\ &= \Sigma_{i=1}^k \alpha_i \Sigma_{j=1}^k \alpha_j l_j(x_i) \\ &= \Sigma_{i,j=1}^k \alpha_i \alpha_j l_j(x_i). \end{aligned}$$

Case (ii). Suppose that $m(\alpha) \in L$ is a superlinear function. Then, one has

$$\begin{aligned} & m(\alpha)(-n(\alpha)) \\ &= m(\alpha)(-\Sigma_{i=1}^k \alpha_i x_i) \\ &\geq \Sigma_{i=1}^k \alpha_i m(\alpha)(-x_i) \\ &= \Sigma_{i=1}^k \alpha_i \Sigma_{j=1}^k \alpha_j l_j(-x_i) \\ &= \Sigma_{i,j=1}^k \alpha_i \alpha_j l_j(-x_i). \end{aligned} \tag{3.10}$$

Since $(x_j,l_j) \in M$ $(j = 1, \cdots, k)$, so by the hypothesis, we have l_j is a sublinear function for each $j = 1, \cdots, k$. Therefore, we conclude that $-l_j(x) \leq l_j(-x)$ for all $x \in X$; $j = 1, \cdots, k$, and hence $-l_j(x_i) \leq l_j(-x_i)$ for all $i,j = 1, \cdots, k$. This, together with (3.10) implies that

$$m(\alpha)(-n(\alpha)) \geq -\Sigma_{i,j=1}^k \alpha_i \alpha_j l_j(x_i).$$

Therefore, in view of case (i), one has

$$\begin{aligned}
&r(\alpha) - m(\alpha)(n(\alpha)) \\
&= \Sigma_{j=1}^{k} \alpha_j l_j(x_j) - m(\alpha)(n(\alpha)) \\
&\geq \Sigma_{j=1}^{k} \alpha_j l_j(x_j) - \Sigma_{i,j=1}^{k} \alpha_i \alpha_j l_j(x_i) \\
&= \Sigma_{i,j=1}^{k} \alpha_i \alpha_j l_j(x_j) - \Sigma_{i,j=1}^{k} \alpha_i \alpha_j l_j(x_i) \\
&= \Sigma_{i,j=1}^{k} \alpha_i \alpha_j [l_j(x_j) - l_j(x_i)] \\
&= \Sigma_{j<i} \alpha_i \alpha_j [l_j(x_j) - l_j(x_i)] + \Sigma_{j>i} \alpha_i \alpha_j [l_j(x_j) - l_j(x_i)] \\
&= \Sigma_{j<i} \alpha_i \alpha_j [l_j(x_j) - l_j(x_i)] + \Sigma_{j<i} \alpha_i \alpha_j [l_i(x_i) - l_i(x_j)] \\
&= \Sigma_{j<i} \alpha_i \alpha_j [l_i(x_i) - l_i(x_j) - l_j(x_i) + l_j(x_j)] \\
&\geq 0,
\end{aligned}$$

because M is an L-monotone set, which proves (3.8). In view of case (ii) and by a similar argument as the above we get (3.9). □

Corollary 3.1. *Let $M \subseteq X \times L$ be a non-empty L-monotone set such that for each $(x, l) \in M$, we have l is a sublinear function. Then*

$$2r(\alpha) + \|n(\alpha)\|^2 + \|m(\alpha)\|^2 \geq 0, \quad \forall\ \alpha \in \mathrm{co}(M).$$

Proof. Since for each $\alpha \in \mathrm{co}(M)$, we have $m(\alpha) \in L$ is a sublinear or superlinear function, then by using Lemma 3.3, we get

$$2r(\alpha) + \|n(\alpha)\|^2 + \|m(\alpha)\|^2 \geq \|n(\alpha)\|^2 + \|m(\alpha)\|^2 + 2m(\alpha)(n(\alpha)) \quad (3.11)$$

for all $\alpha \in \mathrm{co}(M)$ such that $m(\alpha)$ is sublinear, and

$$2r(\alpha) + \|n(\alpha)\|^2 + \|m(\alpha)\|^2 \geq \|n(\alpha)\|^2 + \|m(\alpha)\|^2 - 2m(\alpha)(-n(\alpha)) \quad (3.12)$$

for all $\alpha \in \mathrm{co}(M)$ such that $m(\alpha)$ is superlinear. Therefore, in view of (3.11), (3.12) and Lemma 3.2 we conclude that

$$\begin{aligned}
&2r(\alpha) + \|n(\alpha)\|^2 + \|m(\alpha)\|^2 \\
&\geq \|n(\alpha)\|^2 + \|m(\alpha)\|^2 - 2\|n(\alpha)\|\|m(\alpha)\| \\
&= (\|n(\alpha)\| - \|m(\alpha)\|)^2 \geq 0, \quad \forall\ \alpha \in \mathrm{co}(M),
\end{aligned}$$

which completes the proof. □

Lemma 3.4. *Suppose that Assumption* $(\mathcal{R})$ *holds. Let* $M \subseteq X \times L$ *be a non-empty set. Then* (i) *implies* (ii).

(i)

$$2r(\alpha) + \|n(\alpha)\|^2 + \|m(\alpha)\|^2 \geq 0, \quad \forall\ \alpha \in \mathrm{co}(M).$$

(ii) *There exists* $(x_0, l_0) \in X \times L$ *such that*

$$2[l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0)] \geq \|x_0\|^2 + \|l_0\|^2 + 2l_0(x_0), \quad \forall\ (x_1, l_1) \in M.$$

Proof. First, consider the following assertions:

(iii) There exists $(x_0, l_0) \in X \times L$ such that

$$2r(\alpha) - 2l_0(n(\alpha)) - 2m(\alpha)(x_0) - \|x_0\|^2 - \|l_0\|^2 \geq 0, \quad \forall\ \alpha \in \mathrm{co}(M).$$

(iv) There exists $(x_0, l_0) \in X \times L$ such that

$$2l_1(x_1) - 2l_0(x_1) - 2l_1(x_0) - \|x_0\|^2 - \|l_0\|^2 \geq 0, \quad \forall\ (x_1, l_1) \in M.$$

We show that $(i) \implies (iii) \implies (iv) \implies (ii)$, and hence the result follows.

$(i) \implies (iii)$. Assume that (i) holds. Let $A := \mathrm{co}(M)$ and $F := X \times L$ with the norm:

$$\|(x, l)\| := (\|x\|^2 + \|l\|^2)^{\frac{1}{2}}, \quad (x, l) \in X \times L.$$

It is clear that A is a convex set in $X \times L$, and F is a Banach space. Now, define the function $f : A \longrightarrow \mathbb{R}$ by $f(\alpha) := 2r(\alpha)$ for all $\alpha \in A$, and the function $g : A \longrightarrow F$ by:

$$g(\alpha) := (n(\alpha), m(\alpha))$$

for all $\alpha \in A$. Since n, m and r are linear operators, it follows that f is a convex function and g is an affine function. Then (i) reduces to the following assertion:

$$f(\alpha) + \|g(\alpha)\|^2 \geq 0, \quad \forall\ \alpha \in A. \tag{3.13}$$

In view of Lemma 2.1 we deduce that the assertion (3.13) implies the following assertion:

There exists $y^* \in F^* = L^* \times X^*$ such that

$$f(\alpha) - 2 < g(\alpha), y^* > -\|y^*\|^2 \geq 0, \quad \forall\ \alpha \in A. \tag{3.14}$$

We have $y^* = (l^*, x^*)$ for some $x^* \in X^*$ and $l^* \in L^*$, where X^* is the dual space of X and L^* is the dual space of L. Since Assumption $(\mathcal{R})$ holds, it follows that there exists $x_0 \in X$ such that $l^* = \gamma_{x_0} \in \hat{L}$. By Remark 3.2, we have $\|\gamma_{x_0}\| \leq \|x_0\|$. Assume that $x_0 \neq 0$. Then by Hahn-Banach theorem there exists $l_1 \in X^*$ such that $\|l_1\| = 1$ and $l_1(x_0) = \|x_0\|$. Therefore, since $X^* \subseteq L$, one has $\|x_0\| \leq \|\gamma_{x_0}\|$, and hence $\|\gamma_{x_0}\| = \|x_0\|$. If $x_0 = 0$, then $\|\gamma_{x_0}\| = 0 = \|x_0\|$. Thus, we get

$$\|y^*\| = (\|x_0\|^2 + \|x^*\|^2)^{\frac{1}{2}}. \tag{3.15}$$

Let $l_0 := x^*$. Since $X^* \subseteq L$, we have $l_0 \in L$, and hence $(x_0, l_0) \in X \times L$. Therefore, (3.14) and (3.15) imply (iii).

$(iii) \implies (iv)$. Suppose that (iii) holds. Then, in view of (3.7) we conclude that (iv) follows by restricting $\alpha \in \mathrm{co}(M)$ to the values $\delta_{(x,l)}$.

$(iv) \implies (ii)$. If (iv) holds, then there exists $(x_0, l_0) \in X \times L$ such that

$$2l_1(x_1) - 2l_0(x_1) - 2l_1(x_0) \geq \|x_0\|^2 + \|l_0\|^2, \quad \forall\ (x_1, l_1) \in M. \tag{3.16}$$

By adding $2l_0(x_0)$ to each side of (3.16), we obtain (ii), and the proof is complete. □

Lemma 3.5. *Suppose that Assumption $(\mathcal{R})$ holds. Let $M \subseteq X \times L$ be a non-empty L-monotone set such that for each $(x, l) \in M$, we have l is a sublinear function. Then the following assertions hold:*

(i) There exists $(x_0, l_0) \in X \times L$ such that

$$2[l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0)] \geq \|x_0\|^2 + \|l_0\|^2 + 2l_0(x_0), \quad \forall\ (x_1, l_1) \in M.$$

(ii) Moreover, if M is a maximal L-monotone subset of $X \times L$, then there exists $(x_0, l_0) \in M$ such that

$$\|x_0\|^2 + q^2(-l_0) + 2l_0(x_0) = 0.$$

Proof. (i). Note that (i) is identical with Lemma 3.4 (ii). Therefore, (i) is an immediate consequence of Corollary 3.1 and Lemma 3.4.

(ii). Assume that $(x_0, l_0) \in X \times L$ be as in (i). Then, we have:

$$2[l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0)] \geq 0, \quad \forall\ (x_1, l_1) \in M$$

because by Lemma 3.2, one has $\|x_0\|^2 + \|l_0\|^2 + 2l_0(x_0) \geq 0$. Now, since M is maximal L-monotone, it follows that $(x_0, l_0) \in M$. Thus, by substituting $(x_1, l_1) := (x_0, l_0)$ in (i), we conclude that

$$\|x_0\|^2 + \|l_0\|^2 + 2l_0(x_0) = 0. \tag{3.17}$$

But, by Remark 3.1 we have $\|l_0\| \geq |q(-l_0)|$, and so it follows from (3.17) that

$$\|x_0\|^2 + q^2(-l_0) + 2l_0(x_0) \leq 0.$$

Since by Lemma 3.2, we deduce that

$$\|x_0\|^2 + q^2(-l_0) + 2l_0(x_0) \geq 0,$$

hence the result follows. □

Lemma 3.6. *If $M \subseteq X \times L$ is a maximal L-monotone set and $(0, l_0) \in X \times L$. Then, $M - (0, l_0)$ is a maximal L-monotone set.*

Proof. It is obvious. □

In the sequel, let

$$L_S := \{l \in L : l \text{ is a sublinear function}\}, \tag{3.18}$$

and

$$L_R := \{l \in L : l \text{ is a superlinear function}\}. \tag{3.19}$$

It is easy to check that L_S and L_R are convex cones in L, and $L = L_S \cup L_R$.

Proposition 3.1. *Suppose that Assumption $(\mathcal{R})$ holds. Let $M \subseteq X \times L$ be a maximal L-monotone set such that for each $(x', l') \in M$, we have l' is a sublinear function. Then for every $(0, l) \in \{0\} \times L$ there exists $(x_0, l_0) \in M$ such that*

$$\|x_0\|^2 + q^2(-(l_0 - l)) + 2[(l_0 - l)(x_0)] = 0, \tag{3.20}$$

or

$$\|x_0\|^2 + q^2(-(l_0 + l)) + 2[(l_0 + l)(x_0)] = 0. \tag{3.21}$$

Proof. Assume that M is a maximal L-monotone set such that for each $(x', l') \in M$, we have l' is a sublinear function. Let $(0, l) \in \{0\} \times L$ be arbitrary. Then, $l \in L_R$ or $l \in L_S$. Suppose that $l \in L_R$. Thus for each $(x', l') \in M - (0, l)$, we have l' is a sublinear function. Also, by Lemma 3.6, one has $M - (0, l)$ is a maximal L-monotone set. Therefore, in view of Lemma 3.5 (ii) there exists $(x_1, l_1) \in M - (0, l)$ such that

$$\|x_1\|^2 + q^2(-l_1) + 2l_1(x_1) = 0. \tag{3.22}$$

Since $(x_1, l_1) \in M - (0, l)$, it follows that there exists $(x_0, l_0) \in M$ such that $x_1 = x_0$ and $l_1 = l_0 - l$. Then, in view of (3.22) we get (3.20). If $l \in L_S$, then for each $(x', l') \in M + (0, l)$, we have l' is a sublinear function. Consequently, by a similar argument as the above we obtain (3.21). □

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