ASYMPTOTIC LINEAR ARBITRAGE AND UTILITY-BASED
ASYMPTOTIC LINEAR ARBITRAGE IN MEAN-REVERTING
FINANCIAL MARKETS

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Abstract: Consider a general mean-reverting discrete-time model of financial markets in which the stock prices process is a time discretization of a stochastic differential equation. We introduce a new type of asymptotic arbitrage by proving existence of self-financing strategies that generate linear growing profits on investors’ wealth with probability converging to 1 geometrically fast. We estimate the rate of this convergence using ergodic results on Markov chains and large deviations theory.

Next, we discuss asymptotic linear arbitrage in the expected utility sense and its link with the first type of asymptotic arbitrage.

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1. Introduction

In Mathematical Finance, most models in discrete and continuous-time share the following feature: for any finite time horizon $T < \infty$, there is a possibility to exclude arbitrage opportunity from the market model, see for instance [3]. But, in long-term trading i.e., when $T \to \infty$, one may always generate riskless profit, which is known in the literature as asymptotic arbitrage, see for e.g. the
pioneering works of Kabanov and Kramkov in [7].

This concept of asymptotic arbitrage has been studied since then in slightly different forms by some authors. For examples, after they discussed they subject in a typical case of Urnstein-Uhlenbeck process, Föllmer and Schachermayer in [5] conjectured in a general continuous-time diffusion model the possibility of generating exponential growth profit on investor’s wealth in long-term; what in a corresponding discrete-time setting, we proved in a joint work in [9] and called it “asymptotic exponential arbitrage”. Moreover, we introduced in [9] a more meaningful version of asymptotic exponential arbitrage by considering a general discrete-time stock prices model, expressed in an exponential form and by showing existence of exponential growth profit on investors’ wealth in long-term with the possibility of controlling at a geometrically decaying rate the probability of failing to achieve such a profit.

In this paper, under the modeling settings below, we introduce the new concepts of “asymptotic linear arbitrage with geometrically decaying probability of failure” and “utility-based asymptotic linear arbitrage”. The former is similar to the one we just mentioned above, which we do not recall here as it is indeed similarly defined but treated under different settings.

Consider a discrete-time financial market with two assets in trading: a riskless asset (a bank account or a risk-free bond) with fixed interest rate, set to 0 for simplicity, i.e., with prices normalized to $B_t := 1$ for all time $t \in \mathbb{N}$, and a single risky asset (such as stock) whose (discounted) prices $S_t$, $t \in \mathbb{N}$, is an $\mathbb{R}$-valued process governed by the stochastic difference equation

$$S_{t+1} = S_t + \mu(S_t) + \sigma(S_t)\varepsilon_{t+1}, \quad t \in \mathbb{N}.$$  \hspace{1cm} (1)

$S_0$ is assumed constant, $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, with $\sigma > 0$, are measurable functions determining the drift and volatility of the stock, $(\varepsilon_t)_{t \in \mathbb{N}}$ is an $\mathbb{R}$-valued sequence of i.i.d random variables representing the random driving process of the stock prices evolution. We assume that the stock prices process is modeled and integrable in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{N}}$ with $\mathcal{F}_t := \sigma(S_0, S_1, \ldots, S_t)$, $t \in \mathbb{N}$, is the natural filtration of the stock prices process. $\mathbb{E}$ will always denote the expectation with respect to the probability measure $\mathbb{P}$.

Note that (1) can be thought as the time-discretization of a general diffusion process. In particular, if $\mu(x) := -\alpha x$ with $0 < \alpha < 1$, and $\sigma(x) := 1$ for all $x \in \mathbb{R}$, then we get the discrete-time Ornstein-Uhlenbeck process.

Clearly, the stock prices process $S_t$ in (1) is a (discrete-time) Markov chain in the (uncountable) state space $\mathbb{R}$ (see pp. 211–228 in [1]). Unlike in our joint
work [9] and other similar works such as [3], [5], we do not consider it expressed in any exponential form.

Next, in this market model, trading strategies we consider are \( \mathbb{R} \)-valued \( (\mathcal{F}_t) \)-predictable processes \((\pi_t)_{t \in \mathbb{N}}\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\): \(\pi_t\) represents the number of units of the stock an economic agent holds at each time \(t\) and it is \(\mathcal{F}_{t-1}\)-measurable by predictability. Since since \(\pi_t\) is \(\mathbb{R}\)-valued, then it can be negative; meaning that we allow short-selling of the stock, a more realistic consideration we prohibited from the models in [9].

Given any such trading opportunity \(\pi_t\), we model the corresponding (discounted) wealth an investor allocates in the stock by an \(\mathbb{R}\)-valued discrete-time stochastic process \(V^\pi_t\) obeying the (self-financing) dynamics

\[
\begin{align*}
V^\pi_{t+1} &= V^\pi_t + \pi_{t+1}(S_{t+1} - S_t) \quad \text{for all time } t \geq 1, \\
V^\pi_0 := V_0 &\geq 0, \text{ is the investor’s initial capital.}
\end{align*}
\]

With these modeling settings, we organize the whole paper as follows. In Section 2 below, we define properly the concept of asymptotic linear arbitrage strategies (with geometrically decaying probability of failure). We state in Theorem 2.4 and prove later the main result on existence of such trading strategies in the models (1) and (2) under suitable conditions. And we check that these conditions hold in two practical examples of discrete-time financial models.

Next, in the third and last section of the paper, we also define the concept of “utility-based asymptotic linear arbitrage”, i.e., asymptotic linear arbitrage linked to the concept of expected utilities (see for e.g. [4, Chap. 5]). Classically, an optimal investment for an economic agent with utility function \(U\) is the available portfolio \(\pi_t\) with (random) wealth outcome \(V^\pi_t\) for which the expected utility \(\mathbb{E}U(V^\pi_t)\) is maximal. In that section, we do not focus on the construction of optimal strategies (which is well discussed in the literature, see for e.g. [5]), but rather on treating the following basic question: Among risk-averse and risk-seeking investors, what type of investor (with a suitable utility function) for which if he/her wealth \(V^\pi_t\) grows linearly fast in the sense of Definition 2.2 above, then his/her expected utility will also increase (at least) linearly fast? In Theorem 3.2 we provide an answer to this question for risk-seeking investors with a suitable class of utility functions.

2. The Concept of Asymptotic Linear Arbitrage

**Definition 2.1.** Let \(\pi_t\) be any (self-financing) predictable strategy in the models (1) and (2). We say that \(\pi_t\) is an asymptotic linear arbitrage (ALA)
in the wealth model (2), if from zero initial capital $V_0$, there is a real constant $a > 0$ such that, for all $\epsilon > 0$, there is a time $t_\epsilon \in \mathbb{N}$ satisfying
\[
P(V_t^\pi \geq at) \geq 1 - \epsilon, \text{ for all time } t \geq t_\epsilon.\] (3)

The financial interpretation of this is straightforward: given a “tolerance level” $\epsilon > 0$, there is a threshold time $t_\epsilon$ from which an investor starts to generate profit in long-term at a linear growth rate, with probability tending to 1. However, one may need to wait for a long time $t_\epsilon$ before starting to realize any such profit in long-term. Therefore we formulate a strengthened version of (3) by connecting the tolerance level $\epsilon$ with the running time $t$.

**Definition 2.2.** We say that the trading opportunity $\pi_t$ generates a (strong) asymptotic linear arbitrage (ALA) with geometrically decaying probability of failure (GDP-F) if from zero initial capital $V_0$, there are real constants $a > 0$, and $c > 0$ such that,
\[
P(V_t^\pi \geq at) \geq 1 - e^{-ct} \text{ for all large enough time } t \geq 1,\] (4)
or equivalently,
\[
P(V_t^\pi < at) < e^{-ct}, \text{ for all large enough time } t \geq 1.\] (5)

The additional financial feature of this definition is that, by (4) the investor’s wealth grows linearly fast in long term independently from any threshold time, and by (5) an economic agent can control at a geometrically decaying rate the probability of failing to achieve such a linear growth profit in long-term.

In order to investigate trading strategies $\pi_t$ that generate ALA with GDP-F in the market models (1) and (2):

First, due to the Markovian structure of the stock prices process $S_t$, we restrict ourselves to bounded “Markovian strategies” i.e., trading opportunities of the form $\pi_t := \pi(S_{t-1})$, for all $t \in \mathbb{N}$, where $\pi : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function with respect to the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$.

Next, we assume that the stock prices process $S_t$ in (1) satisfies the so-called “mean-reverting” condition:
\[(MRC) : \limsup_{|x| \to \infty} \frac{|x + \mu(x)|}{|x|} < 1.\] (6)

This is obviously verified for e.g., by the (discrete-time) Ornstein-Uhlenbeck process. It means the stock prices have at most a linear growth and they tend to move about the average price in time.
And we suppose that the following set of conditions holds in the same models (1) and (2):

(A1) The random variables \( \varepsilon_t \)'s have a common (a.s.) strictly positive density \( \gamma \) with respect to the Lebesgue measure \( \lambda \) on \( \mathbb{R} \), and this density is (a.s.) bounded on each compact in \( \mathbb{R} \).

(A2) The drift \( \mu \) is locally bounded. The volatility \( \sigma \) is positive, bounded away from zero on each compact and is (globally) bounded.

(A3) And we assume the following integrability property for the law of the \( \varepsilon_t \)'s:

\[
\exists \kappa > 0 \text{ such that } \mathbb{E}(e^{\kappa \varepsilon^2}) =: I < \infty, \tag{7}
\]

where \( \varepsilon \) has the distribution as the \( \varepsilon_t \)'s, \( t \in \mathbb{N} \). We also assume that \( \mathbb{E}\varepsilon = 0 \) holds\(^1\).

Under these conditions, to proceed to the statement of the existence theorem, we express first the prices process \( S_t \) of the stock from (1) in the form

\[
S_{t+1} - S_t = \mu(S_t) + \sigma(S_t)\varepsilon_{t+1} = \sigma(S_t)(\varphi(S_t) + \varepsilon_{t+1}),
\]

where the function \( \varphi \) is defined by \( \varphi(x) := \mu(x)/\sigma(x) \), for all \( x \in \mathbb{R} \).

**Definition 2.3.** We call \( \varphi \) the “market price of risk” function for the stock prices \( S_t \).

The quantity \( \varphi(S_t) \) bears a straightforward interpretation: since \( \mu(S_t) \) represents the average one-step return of the stock while \( \sigma(S_t) \) measures the one-step volatility of this price as driven by the random “noise” \( \varepsilon_t \), \( \varphi(S_t) \) represents the one-step return of stock per unit volatility.

We may also require the condition below, called “risk-condition”, for the market price of risk function \( \varphi \):

\[
(RC): \text{ the set } R_0 := \{ x \in \mathbb{R} \mid \varphi(x) \neq 0 \} \text{ satisfies } \lambda(R_0) > 0. \tag{8}
\]

We interpret the set \( R_0 \) as representing all states of the stock prices \( S_t \) whose market price of risk is not 0. We say that \( (RC) \) holds if \( \lambda(R_0) > 0 \).

Set \( R_0^+ := \{ x \in \mathbb{R} \mid \varphi(x) > 0 \} \) and \( R_0^- := \{ x \in \mathbb{R} \mid \varphi(x) < 0 \} \). If \( 1_A \) denotes the indicator function on \( A \) for any \( A \subseteq \mathbb{R} \), consider the bounded Markovian strategy

\[
\pi^0_t := \pi^0(S_{t-1}) \text{ where } \pi^0(x) := 1_{R_0^+}(x) - 1_{R_0^-}(x) \text{ for all } x \in \mathbb{R}, \tag{9}
\]

\(^1\)This is not a restriction of generality. If one had \( \mathbb{E}\varepsilon = m \) one could replace \( \mu(x) \) by \( \mu'(x) := \mu(x) + \sigma(x)m \) and \( \varepsilon_t \) by \( \varepsilon_t := \varepsilon_t - m \) and in this way we get back to the case \( \mathbb{E}\varepsilon = 0 \).
which is interpreted as being constructable by a potential long-term arbitrageur as follows: s/he invests all his money in the stock whenever its market price of risk is positive, s/he sells the stock short when the market price of risk is negative, otherwise he puts everything into his bank account. Then we state the first main result of this paper.

**Theorem 2.4.** Suppose that the market price of risk function $\varphi$ satisfies the risk-condition (RC) in (8). Then the bounded Markovian strategy $\pi^0_t = 1_{R^+}(S_{t-1}) - 1_{R^{-}}(S_{t-1})$ generates an ALA with GDP-$F$ in the models (1) and (2).

We present the proof at the end of this section after an appropriate preparation.

First, inspecting the dynamics of the investor’s wealth process in (2) for any bounded Markovian strategy $\pi_t = \pi(S_{t-1})$, we express it in the functional form

$$V^\pi_t = V_0 + \sum_{n=1}^t f(\Phi_n), \text{ for all time } t \geq 1,$$

of the auxiliary process $\Phi_n := (S_{n-1}, S_n)$ of two consecutive values of the stock prices process, where $f$ is the measurable function defined on $\mathbb{R}^2$ by $f(x, y) := \pi(x)(y - x)$. Assume that $S_{-1}$ is an (arbitrary) given initial constant so that the process $\Phi_t = (S_{t-1}, S_t)$ starts at time 0 as well.

We show below a first set of preliminary results derived from the advanced theory of Markov chains presented in [10] and from the ergodic theory of functionals of Markov chains in [8].

**Proposition 2.5.** The stochastic process $\Phi_t$ is a Markov chain with state space $\mathbb{R}^2$.

**Proof.** We derive this from [1] pp. 211-228, where the Markov property of any (discrete-time) Markov chain $Y_t$ in a Polish state space $S$ is characterized by its evolution in the form $Y_{t+1} = f(Y_t, \xi_{t+1})$, with $(\xi_t)$ a sequence of i.i.d random variables independent of $Y_0$ and valued in some measurable space $S'$, and $f : S \times S' \to S$ a suitable measurable function. Indeed, using (1), the Markov chain $S_t$ is in the form $S_{t+1} = f(S_t, \varepsilon_{t+1})$ for all time $t \in \mathbb{N}$, with the measurable function $f$ defined by $f(x, y) := x + \mu(x) + \sigma(x)y$ for all $x, y \in \mathbb{R}$. It follows for all time $t \in \mathbb{N}$, that we have $\Phi_{t+1} = (S_t, S_{t+1}) = (S_t; f(S_t, \varepsilon_{t+1})) = F(\Phi_t, \xi_{t+1})$, where $\xi_t := (0, \varepsilon_t)$ and $F$ is the measurable function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by
F((x, y); (a, b)) := (y; f(y, b)). Since the εt’s are i.i.d and independent from $S_0$, the ξt’s are also i.i.d and independent from $\Phi_0$, showing that the next state $\Phi_{t+1}$ is generated from the previous state $\Phi_t$, plus an independent noise $\xi_{t+1}$, as required. □

Let $\lambda_2$ denote the Lebesgue measure on $\mathbb{R}^2$ and $\mathcal{B} (\mathbb{R}^2)$ the Borel $\sigma$-algebra on $\mathbb{R}^2$. For $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$, let us denote $P(x, A) := \mathbb{P}(S_{t+1} \in A | S_t = x)$, $t \geq 0$, the one-step transition probability kernel of the chain $S_t$, and $P^t(x, A) := \mathbb{P}(S_t \in A | S_0 = x)$ its $t$-step transition probability kernel. Also for $z \in \mathbb{R}^2$ and $C \in \mathcal{B}(\mathbb{R}^2)$, let $Q(z, C)$ and $Q^t(z, C)$ denote the corresponding kernels for the chain $\tilde{\Phi}_t$. Then we have the following

**Proposition 2.6.** The Markov chain $S_t$ is $\psi$-irreducible and aperiodic.

**Proof.** By the definition of $\psi$-irreducibility and Propositions 4.2.1 and 4.2.2 on pp. 89-90 in [10], it is enough to show it is $\lambda$-irreducible, that is; if $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$ such that $\lambda(A) > 0$, then, there is an integer $t \geq 1$ such that $P^t(x, A) > 0$. This clearly follows for $t = 1$ by Assumption ($A_1$) and by the translation invariance property of the Lebesgue measure $\lambda$ applied in the last of the equalities below:

$$
P(x, A) := \mathbb{P}(S_{t+1} \in A | S_t = x) = \mathbb{P}(x + \mu(x) + \sigma(x)\varepsilon_{t+1} \in A) = \int_{(A-x-\mu(x))/\sigma(x)} \gamma(y) \lambda(dy) > 0. \quad (11)
$$

Next, to prove the aperiodicity of $S_t$, by Definition 5.14 on p.109 in [10] and by Theorem 5.4.4 on p.121 of the same reference, it is enough to show that there exists a subset $C$ in $\mathbb{R}$ with $\lambda(C) > 0$, $n > 0$ and a non-trivial measure $\nu_n$ on $\mathcal{B}(\mathbb{R})$ such that

$$
P^n(x, A) \geq \nu_n(A) \text{ for all } x \in C, A \in \mathcal{B}(\mathbb{R}), \quad (12)
$$

and the $\text{g.c.d}$ (greatest common divisor) of the set $E_C$ is 1, where

$$
E_C := \{ n \geq 1 : C \text{ satisfies } (12) \}.
$$

Indeed, fix any compact subset $C$ in $\mathbb{R}$ with $\lambda(C) > 0$ and let $n := 1$. Using Assumptions ($A_1$) and ($A_2$), it follows from (11) that $P^1(x, A) = P(x, A) \geq c\lambda 1_C(A)$ for some constant $c > 0$ and for all $x \in C$, $A \in \mathcal{B}(\mathbb{R})$. Indeed,

$$
c := (\sup \{ \sigma(u) : u \in C \})^{-1} \inf \left\{ \gamma \left( \frac{a-u-\mu(u)}{\sigma(u)} \right) : a \in C, u \in C \right\}.
$$

So, taking $\nu_1 := c\lambda 1_C$, where $\nu_1(dy) := c\lambda(dy \cap C)$, we get $1 \in E_C$, hence $\text{g.c.d}(E_C) = 1$. □
Proposition 2.7. The Markov chain $\Phi_t$ is also $\psi$-irreducible and aperiodic.

Proof. Similarly to the preceding proof, we show first that $\Phi_t$ is irreducible. Using condition $(A_1)$, for all $y \in \mathbb{R}$ the random variable $y + \mu(y) + \sigma(y)\varepsilon_1$ has a $\lambda$-a.e. positive density, $p_1(u), u \in \mathbb{R}$. By the same argument, for all $u \in \mathbb{R}$ the random variable $y + \mu(y) + \sigma(y)u + \sigma(y)\mu(y)u + \sigma(y)\mu(y)\varepsilon_2$ has a $\lambda$-a.e. positive density $p_2(u, w), w \in \mathbb{R}$ which can be chosen jointly measurable in $(u, w)$. Hence, by independence of $\varepsilon_1, \varepsilon_2$, when $\Phi_0 = (x, y)$, the density of

$$\Phi_2 = (y + \mu(y) + \sigma(y)\varepsilon_1, y + \mu(y) + \sigma(y)\varepsilon_1 + \sigma(y)\mu(y)\varepsilon_2)$$

with respect to $\lambda_2$ equals $p_1(u)p_2(u, w)$, and this is $\lambda_2$-a.e. positive. In particular, for all $A, B \in \mathcal{B}(\mathbb{R})$ with $\lambda_2(A \times B) > 0$, setting $(x, y) = z$, we have

$$Q^2(z, A \times B) = \mathbb{P}(\Phi_2 \in A \times B | \Phi_0 = z) = \int_{A \times B} p_1(u)p_2(u, w)\lambda_2(du, dw),$$

(13)

which is strictly positive, showing $\lambda_2$-irreducibility and hence $\psi$-irreducibility of the Markov chain $\Phi_t$.

Next for aperiodicity, take any compact rectangle $C := C_1 \times C_2$ such that $\lambda_2(C) > 0$ with $C_1$ and $C_2$ intervals in $\mathbb{R}$, there exist constants $c_1, c_2 > 0$ such that, with the measure $\nu_2 := c_1c_2\lambda_21_C$ defined on $\mathcal{B}(\mathbb{R}^2)$ by $\nu_2(dy_1, dy_2) = c_1c_2\lambda_2(dy_1 \cap C_1, dy_2 \cap C_2)$, we have $Q^2((x, y); A \times B) \geq \nu_2(A \times B)$ for all $x \in C_1, y \in C_2$ and all $A, B \in \mathcal{B}(\mathbb{R})$, which proves that $2 \in E_C$ (where $E_C$ is defined using the kernel $Q$ instead of $P$ in (12)). Moreover, by an additional similar argument, one gets also $3 \in E_C$, showing that $g.c.d(E_C) = 1$, as required. \[ \square \]

Lemma 2.8. The random variable $\varepsilon$ in (7) of Assumption $(A_3)$ satisfies the following property: for every real number $a \geq 1$, we have,

$$\mathbb{E}(e^{a|\varepsilon|}) \leq e^{ca^2},$$

(14)

for some fixed constant $c > 0$.

Proof. Set $\xi := |\varepsilon|$. Then we have

$$\mathbb{P}(e^{a\xi} > x) = \mathbb{P}(\exp\left(\kappa\left[\frac{\log(e^{a\xi})}{a}\right]^2\right) > \exp\left(\kappa\left[\frac{\log x}{a}\right]^2\right)) \leq I \exp\left(-\kappa\left(\log(x)/a\right)^2\right)$$

by Markov Inequality

$$= I(\frac{1}{x})(\kappa/a^2)\log x,$$

see (7) for the definition of $I$. Since the exponent $(\kappa/a^2)\log x > 2$ provided that $x > e^{2a^2/\kappa}$, we have

$$\mathbb{E}(e^{a\xi}) = \int_0^\infty \mathbb{P}(e^{a\xi} > x)dx \leq e^{2a^2/\kappa} + I\int_{\exp(2a^2/\kappa)}^\infty 1/x^2dx.$$
The last integral is less than $\int_1^{\infty} 1/x^2 \, dx$, which is finite, thus we conclude the proof by taking for example $c = c_1 + (2/\kappa)$ with $c_1 > 0$ large enough.

Applying this lemma, we check below the following

**Proposition 2.9.** The Markov chain $S_t$ satisfies the “drift condition” $(DV3+)(i)$ on p. 6 of [8], which is recalled in the proof below.

Proof. $(DV3+)(i)$ means: the chain $S_t$ is $\psi$-irreducible, aperiodic and there are (measurable) functions $V, W : \mathbb{R} \to [1, \infty)$, a subset $C$ in $\mathbb{R}$ verifying (12) above for some $n$, and constants $\delta > 0, b < \infty$ such that $\log(e^{-V} P e^V)(x) \leq -\delta W(x) + b 1_C(x)$, for all $x \in \mathbb{R}$, with $P e^V$ defined by

$$P e^V (x) := \int e^V(y) P(x, dy), \text{ for all } x \in \mathbb{R}.$$ 

This is equivalent to requiring that,

$$P e^V (x) \leq e^{V(x) - \delta W(x) + b 1_C(x)} \text{ for all } x \in \mathbb{R}. \quad (15)$$

By Proposition 2.6, $S_t$ is $\psi$-irreducible and aperiodic. Next, define $V(x) = W(x) := 1 + qx^2$, for all $x \in \mathbb{R}$, where $q > 0$ is a small number to be chosen. Consider a compact set $C := [-K, K]$ for a large positive constant $K$. As in the proof of Proposition 2.6, $C$ satisfies (12).

Since $P e^V (x) = \mathbb{E}(e^{V(S_1)} | S_0 = x) = \mathbb{E}(e^{V(x+\mu(x)+\sigma(x)\varepsilon)})$, it follows from (15) that we need to show for all $x \in \mathbb{R}$,

$$\mathbb{E}(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}) \leq e^{(1-\delta)V(x)+b 1_C(x)}. \quad (16)$$

To get this, it is sufficient to prove the two conditions below:

**Claim 1:** for $|x|$ large enough we have

$$\mathbb{E}(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}) \leq e^{(1-\delta)(1+qx^2)}. \quad (17)$$

**Claim 2:** for small $|x|$, (that is; for $x$ in any fixed compact), we have

$$\sup_{x \in C} \mathbb{E}(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}) < G(K), \quad (18)$$

for some positive constant $G(K) < \infty$, and then once this is done, take $b := \log G(K)$.

**Proof of Claim 1.** Using the mean-reverting condition $(MRC)$ in (6), for $|x|$ large enough, there is a small $\delta > 0$ such that we have $(x + \mu(x))^2 \leq
\( (1-4\delta)x^2 \). And since \( 1 \leq \delta(1+q x^2) \) for \( |x| \) large, it follows that \( e^{1+q(x+\mu(x))^2} \leq e^{(1-3\delta)(1+q x^2)} \).

By \((A_2)\) there is \( M > 0 \) such that, for all \( x, \sigma(x) \leq M \). If we choose \( q \) such that \( qM^2 < \kappa/2 \), then it is enough to show that \( \mathbb{E}(e^{2q|x+\mu(x)|M|x|+(\kappa/2)x^2}) \leq e^{2\delta x^2} \). By the Cauchy-Schwarz inequality, this requires to prove that,

\[
\sqrt{\mathbb{E}(e^{4q|x+\mu(x)||M|x|})} \sqrt{\mathbb{E}(e^{\kappa x^2})} \leq e^{2\delta x^2} \tag{19}
\]

By \((7)\), the second term on the left-hand side of \((19)\) is the constant \( \sqrt{T} \). This is smaller than \( e^{\delta x^2} \) for large enough \( |x| \). So, since again by Condition \((MRC)\), \( 4q|x+\mu(x)|M \leq 4qM|x| \) for \( |x| \) large, it follows that we finally have to show \( \sqrt{\mathbb{E}(e^{4qM|x||x|})} \leq e^{\delta x^2} \) for large \( |x| \), or, equivalently,

\[
\mathbb{E}(e^{4qM|x||x|}) \leq e^{2\delta x^2} \text{ for large } |x|. \tag{20}
\]

But applying Lemma 2.8, the left-hand side of \((20)\) is smaller than \( e^{16c_0^2M^2|x|^2} \) for some fixed constant \( c > 0 \). Hence, if one chooses \( q \) small enough such that \( 16q^2M^2c < 2\delta \) and \( qM^2 < \kappa/2 \) then \((20)\) holds, showing Claim 1.

**Proof of Claim 2.** By Assumption \((A_2)\), \( \mu \) is bounded above on any compact \( C = [-K, K] \) by some positive constant \( A \). Since \( \mu \) is bounded on \( C \), the function \( x \mapsto (x + \mu(x))^2 \) is also bounded on \( C \). We assume it bounded above on that \( C \) by some positive constant \( B \). So, with the later choice of \( q \), we have the following estimate applying Cauchy-Schwarz Inequality and \((7)\),

\[
\mathbb{E}(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)e+q\sigma^2(x)e^2}) \leq e^{1+qB+2q(K+A)M|x|+(\kappa/2)x^2}) \leq e^{(1+qB)} \sqrt{\mathbb{E}(e^{4q(K+A)M|x|})} \sqrt{\mathbb{E}(e^{\kappa x^2})} = e^{(1+qB)} \sqrt{T} \sqrt{\mathbb{E}(e^{4q(K+A)M|x|})}.
\]

We then choose \( K \) large enough such that \( 4q(K+A)M \geq 1 \) and we get, by Lemma 2.8, that for all \( x \in C = [-K, K] \),

\[
\mathbb{E}(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)e+q\sigma^2(x)e^2}) \leq e^{(1+qB)} \sqrt{T} \sqrt{e^{16c_0^2(K+A)^2M^2}},
\]

for a fixed constant \( c' > 0 \). This holds for all \( x \in C \), hence \((18)\) holds true when taking the supremum over \( C \) of the left-hand side of this latter inequality. This completes the proof of the whole proposition.

**Corollary 2.10.** The Markov chain \( \Phi_t = (S_{t-1}, S_t) \) also satisfies the drift condition \((DV3+)\) (i) on p. 6 of [8].
Proof. This follows from the preceding proposition and Proposition 4.1 (v) of [8]. □

Furthermore,

**Proposition 2.11.** The Markov chain \( \Phi_t = (S_{t-1}, S_t) \) also satisfies the second condition (ii) of (DV3+) on p.6 of [8], recalled in the proof below.

Proof. We have to show that: there are functions \( V, W : \mathbb{R}^2 \rightarrow [1, \infty) \), and there is a time \( t_0 > 0 \) such that for all \( r < ||W||_{\infty} \), there is a measure \( \beta_r \) such that \( \beta_r(e^V) := \int_{\mathbb{R}^2} e^{V(x,y)} \beta_r(dx, dy) \) is finite and we have

\[
Q_{(x,y)} \left( \Phi_{t_0} \in A \times B, \tau_{C_W^c(r)}>t_0 \right) \leq \beta_r(A \times B),
\]

for all \( (x, y) \in C_W(r) := \{(x, y) : W(x, y) \leq r\} \) and all \( A, B \in B(\mathbb{R}) \). Here \( \tau_{C_W^c(r)} := \min\{t \geq 1 : \Phi_t \in C_W^c(r)\} \) and \( C_W^c(r) \) denotes the complement of \( C_W(r) \).

Consider then the functions \( V(x, y) = W(x, y) := 1 + q(x^2 + y^2) \) for \( x, y \in \mathbb{R} \) for a suitable \( q > 0 \) as in the proof of Proposition 2.9. Since the chain \( \Phi_t \) starts at time \( t = 1 \), we choose here \( t_0 := 2 \), and let \( r < ||W||_{\infty} = \infty \). Then:

If \( 0 \leq r < 1 \), the statement below logically follows.

Suppose \( r \geq 1 \), we have \( C_W(r) = \{(x, y) : 1 + q(x^2 + y^2) \leq r\} = \{(x, y) : x^2 + y^2 \leq (r - 1)/q\} \) which is the compact disk of radius \((r - 1)/2\) in \( \mathbb{R}^2 \). So its first and second projections \( C_1 := pr_1(C_W(r)) \) and \( C_2 := pr_2(C_W(r)) \) are compact intervals in \( \mathbb{R} \).

If \( A, B \in B(\mathbb{R}) \) and \( (x, y) \in C_W(r) \), then \( x \in C_1 \) and \( y \in C_2 \). Hence, setting for simplicity \( \Delta := Q_{(x,y)}(\Phi_2 \in A \times B, \tau_{C_W^c(r)}>2) \), we obtain that,

\[
\Delta = P(\Phi_2 \in A \times B, \Phi_2 \in C_W(r) \mid \Phi_1 = (x, y)) = P(\Phi_2 \in (A \times B) \cap C_W(r) \mid \Phi_1 = (x, y)) \leq P(x + \mu(x) + \sigma(x)e_0 \in A \cap C_1, y + \mu(y) + \sigma(y)e_1 \in B \cap C_2) \leq J_r J'_r \lambda(A \cap C_1) \lambda(B \cap C_2) =: \beta_r(A \times B),
\]

for some constants \( J_r \) and \( J'_r \) (depending on \( q \)), using the independence of \( e_0 \) and \( e_1 \), Fubini’s Theorem and Assumptions \((A_1)\), \((A_2)\). \( \beta_r \) so defined is clearly a measure on \( \mathbb{R}^2 \).
Finally, it is clear that $\beta_r(e^V) = \int_{C_1 \times C_2} J_r J'_r e^{1+q(x^2+y^2)} \lambda_2(dx,dy) < \infty$ as integral of a continuous function on a compact of $\mathbb{R}^2$ with respect to $\lambda_2$, ending the proof.

Corollary 2.12. The Markov chain $\Phi_t$ has an invariant probability measure $\nu$ equivalent to the Lebesgue measure $\lambda_2$ on $\mathbb{R}^2$.

Proof. By Corollary 2.10 and Proposition 2.11 above, $\Phi_t$ satisfies the whole condition $(DV3+) (i)$ and $(ii)$. It follows by Theorem 1.2 in [8] that the Markov chain $\Phi_t$ has a unique invariant probability measure, say $\nu$.

Moreover, from the proof of $\psi$-irreducibility of $\Phi_t$ in Proposition 2.7, $\mathbb{P}(\Phi_2 \in \cdot | \Phi_0 = (x,y))$ is $\lambda_2$-absolutely continuous for each $(x,y) \in \mathbb{R}^2$, hence we get $\nu \ll \lambda_2$. On the other hand, since the chain $\Phi_t$ is $\psi$-irreducible with $\nu$ as its invariant probability measure, from the definition of recurrent and positive chains on pp. 186 and 235 of [10], it follows by Proposition 10.1.1 and Theorem 10.4.9 of the same reference that $\nu \sim \psi$. But, $\psi \gg \lambda_2$ by Proposition 4.2.2 $(ii)$ in [10], so $\nu \gg \lambda_2$, and hence $\nu \sim \lambda_2$, as required.

Next, after this first set of preliminary results, as indicated in the introduction, we now proceed to the application of classical large deviations techniques from [2]. First, recall the investor’s wealth process as in (10) for any given Markovian strategy $\pi_t = \pi(S_{t-1})$,

$$V_0^\pi = V_0 + \sum_{n=1}^t f(\Phi_n), \text{ for all time } t \geq 1,$$

with $f(x,y) = \pi(x)(y-x)$, for all $x, y \in \mathbb{R}$.

We have to insure that, for every $\pi_t$, the sequence of random variables $(V_0^\pi - V_0)_t = \sum_{n=1}^t f(\Phi_n)$ satisfies the LDP hypotheses, that is; the limit $\Lambda_f(\theta) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{\theta \sum_{n=1}^t f(\Phi_n)})$ for each $\theta \in \mathbb{R}$, exists with $\Lambda_f$ satisfying the remaining conditions in Gärtner-Ellis’ theorem as stated in Theorem 2.3.6, [2].

For that, since we obtained from Corollary 2.10 and Proposition 2.11 that the Markov chain $\Phi_t$ is $\psi$-irreducible, aperiodic and satisfies the conditions $(DV3+) (i)$, $(ii)$ in [8], we need to apply ergodic results related to $\Phi_t$ from that article. We adopt the notations of [8] below and before checking the conditions under which these results hold. Indeed, considering the functions $V, W$ in the proof of Proposition 2.11, define another function $W_0$ by $W_0(x,y) := 1 + q(|x| + |y|)$, for $x, y \in \mathbb{R}$, where $q$ is the same as in the definition of $V, W$ in
that proposition, we obviously see that
\[
\lim_{r \to \infty} \sup_{x,y \in \mathbb{R}} \left( \frac{W_0(x,y)}{W(x,y)} 1_{W(x,y)>r} \right) = 0,
\]
which is Condition (6) on p. 7 of [8].

For every \( \theta \in \mathbb{R} \), we observe that
\[
\theta \sum_{t=1}^{n} f(\Phi_n) = \sum_{t=1}^{n} F_{\theta}(\Phi_n),
\]
where \( F_{\theta} = \theta f \). Consider the Banach space \( L_{W_0}^{\infty} \) defined on p. 4 of [8] by
\[
L_{W_0}^{\infty} := \{ h : \mathbb{R}^2 \to \mathbb{C} : \sup_{x,y} |h(x,y)|/W_0(x,y) < \infty \},
\]
which is equipped with the norm \( \| h \|_{W_0} := \sup_{x,y} |h(x,y)|/W_0(x,y) \), for \( h \in L_{W_0}^{\infty} \). Then we have the following

**Lemma 2.13.** For all \( \theta \in \mathbb{R} \), the function \( F_{\theta} \) belongs to the space \( L_{W_0}^{\infty} \).

**Proof.** It is enough to show this for \( \theta = 1 \). Indeed, by the boundedness assumption of \( \pi \), for some constant \( c > 0 \), we have \(|\pi(x)| \leq c|y-x| \) for all \( x \in \mathbb{R} \). If follows that \(|F_1(x,y)| \leq c|y-x| \) for all \( x, y \in \mathbb{R} \). Since clearly \(|y-x| \leq 1 + |x| + |y|\), then we obtain that \(|F_1(x,y)| \leq c(1 + |x| + |y|)\), for all \( x, y \in \mathbb{R} \). Hence, taking the supremum over \((x, y) \in \mathbb{R}^2\), we get \(\sup_{x,y} |F_1(x,y)|/W_0(x,y) < \infty\); that is \( F_1 \in L_{W_0}^{\infty} \), as required. \( \square \)

Next, consider the sequence of non-linear operators \( \Gamma_t : L_{W_0}^{\infty} \to L_{V}^{\infty} \) defined as in [8], by setting for all \( F \in L_{W_0}^{\infty} \) and all \((x, y) \in \mathbb{R}^2\),
\[
\Gamma_t(F)(x, y) := \frac{1}{t} \log \mathbb{E}_{x,y} \left( \exp \left( \sum_{n=1}^{t} F(\Phi_n) \right) \right).
\]
(23)

where \( \mathbb{E}_{x,y} \) means that we have started the chain from \( \Phi_0 := (x, y) \) and we compute the expectation accordingly. Then we get,

**Proposition 2.14.** Let \( \pi_t \) be any bounded Markovian strategy in the model (2). Then there is an analytic function
\[
\Lambda_f(\theta) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_{-1},S_0) \left( e^{\theta(V^\pi_t-V_0)} \right),
\]
defined for all \( \theta \in \mathbb{R} \), such that the average sum \((V^\pi_t-V_0)/t \) satisfies an LDP (large deviations principle) with good convex rate function \( \Lambda_f^* \) (the convex conjugate of \( \Lambda_f \)).

**Proof.** Since \( \Phi_t \) satisfies the conditions \((DV3+)\) (i), (ii) in [8] with the previous unbounded \( W \), then by Proposition 3.6, [8], there is a non-linear operator
\( \Gamma : L^W_{\infty} \rightarrow L^V_{\infty} \) such that the following uniform convergence holds over balls in \( L^W_{\infty} \):

\[
\sup_{\|F - F_0\|_{W_0} \leq \delta} \|\Gamma_t(F) - \Gamma(F)\|_V \rightarrow 0 \text{ as } t \rightarrow \infty,
\]

for each \( F_0 \) and each \( \delta > 0 \). For every \( F_0 \) and each \( \delta > 0 \), for each \( \theta \in \mathbb{R} \), set \( F_\theta := F_\theta = \theta g \) and \( F_0 := 0 \). Since \( V_t^\pi \) depends on \( g \), it follows that for all \( \theta \in \mathbb{R} \), the limit

\[
\Lambda_f(\theta) := \Gamma(F_\theta)(S_{-1}, S_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_{(S_{-1}, S_0)} \left( \exp \left( \sum_{n=1}^{t} \theta f(\Phi_n) \right) \right)
\]

exists in \( \mathbb{R} \). Moreover, applying Proposition 4.3 (ii) in [8], \( \Lambda_f \) is an analytic function of \( \theta \).

Again from (ii) of Proposition 4.3, [8], we deduce the second-order Taylor expansion about zero as, \( \Lambda_f(\theta) = \Lambda_f(0) + \theta \nu(f) + \frac{1}{2} \theta^2 v_f + O(\theta^3) \) for all \( \theta \in \mathbb{R} \), where \( \nu \) is the invariant measure of \( \Phi_t \) obtained in Corollary 2.12, the expectation \( \nu(f) := \int_{\mathbb{R}^2} f(x, y) \nu(dx, dy) \) is finite, and where \( v_f := \lim_{t \rightarrow \infty} \mathbb{E} \nu \sum_{n=1}^{t} (f(\Phi_n) - \nu(f))^2 \neq 0 \) is the asymptotic variance given in (37), p. 24 of [8]. Hence \( \Lambda_f(\theta) \) is essentially smooth.

So, applying Gärtner-Ellis Theorem 2.3.6 in [2], we conclude that \( (V_t^\pi - V_0)/t \) satisfies an (upper) LDP estimate with good convex rate function \( \Lambda_f^* \), as we required.

**Proposition 2.15.** Under the conditions of the preceding proposition, \( \nu(f) \) is the unique minimizer of \( \Lambda_f^* \). Moreover \( \Lambda_f^*(x) > 0 \) for all \( x \neq \nu(f) \).

**Proof.** Using (24), we see that \( \Lambda_f(0) = 0 \). And from the Taylor expansion of \( \Lambda_f \) in the preceding proof, we have \( \Lambda_f'(0) = \nu(f) \), so we get by Lemma 2.4 of [6] that \( \Lambda_f^*(\nu(f)) = \nu(f) \times 0 - \Lambda_f(0) = 0 \). On the other hand, by definition of a conjugate function, we always have \( \Lambda_f^*(x) \geq 0 \times x - \Lambda_f(0) = 0 \) for all \( x \in \mathbb{R} \). It follows that \( \nu(f) \) is a global minimizer for \( \Lambda_f^* \). Since by Proposition 2.14 above, \( \Lambda_f \) is analytic hence differentiable, it follows that its conjugate \( \Lambda_f^* \) is strictly convex on its effective domain which is, in fact, \( \mathbb{R} \). This implies that the global minimizer \( \nu(f) \) for \( \Lambda_f^* \) is unique. And this uniqueness implies that \( \Lambda_f^*(x) > 0 \) for all \( x \neq \nu(f) \), as required.

Finally, before the proof of the main theorem, we give the following
Proposition 2.16. Suppose that the market price of risk function $\varphi$ in Definition 2.3 satisfies the risk-condition $(RC)$ set in (8). Then the bounded Markovian strategy $\pi^0_t$ constructed in (9) satisfies

$$\nu(f) = \mathbb{E}(\pi^0_t(S_0)(S_1 - S_0)) > 0,$$

where $(\tilde{S}_0, \tilde{S}_1)$ has distribution $\nu$, the invariant probability measure of $\Phi_t$.

Proof. Since $\nu$ is a probability measure on $\mathcal{B}(\mathbb{R}^2)$ and is invariant for the Markov chain $\Phi_t = (S_{t-1}, S_t)$, then there is a pair of $\mathbb{R}$-valued random variables $(\tilde{S}_0, \tilde{S}_1 = \tilde{S}_0 + \mu(\tilde{S}_0) + \sigma(\tilde{S}_0)\varepsilon_1)$ on $\Omega$ with distribution $\nu$ and such that $\varepsilon_1$ is still independent of $\tilde{S}_0$. For all $x \in \mathbb{R}$,

$$\mathbb{E}(\tilde{S}_1 | \tilde{S}_0 = x) = \mathbb{E}(x + \mu(x) + \sigma(x)\varepsilon_1 | \tilde{S}_0 = x) = x + \mu(x) + \sigma(x)\mathbb{E}(\varepsilon_1 | \tilde{S}_0 = x) = x + \sigma(x)\varphi(x)$$

by independence of $\varepsilon_1$ from $\tilde{S}_0$.

Since $(A_2)$ implies $\sigma > 0$, it follows that if $x \in R_0$ (the set defined in (8)), then we have

$$\mathbb{E}(\tilde{S}_1 | \tilde{S}_0 = x) \neq x. \quad (26)$$

Consider our constructed strategy $\pi^0_t$ given by the function

$$\pi^0_t := 1_{R_0^c}(x) - 1_{R_0}(x), \text{ for all } x \in \mathbb{R}.$$  

By Corollary 2.12, $\nu$ has a $\lambda_2$-a.e. positive density with respect to $\lambda_2$, hence its $\tilde{S}_0$-marginal, denoted by $\eta$, has a $\lambda$-a.e. positive density $\ell(x)$. Therefore,

$$\nu(f) = \int_{\mathbb{R}} \mathbb{E}(\pi^0_t(x)(\tilde{S}_1 - x) | \tilde{S}_0 = x)\eta(dx) = \int_{R_0} \mathbb{E}(\tilde{S}_1 - x | \tilde{S}_0 = x)\ell(x)\lambda(dx) = \int_{R_0} \text{sgn}(\mathbb{E}(\tilde{S}_1 - x | \tilde{S}_0 = x))\mathbb{E}(\tilde{S}_1 - x | \tilde{S}_0 = x)\ell(x)\lambda(dx),$$

which is strictly positive. Hence $\nu(f) > 0$, showing the result. \hfill $\square$

Proof of Theorem 2.4. Proposition 2.14 says that the average sum $(V_t^{\pi^0} - V_0)/t$ satisfies an LDP with good rate function $\Lambda^*_f$ and Proposition 2.16 above says $\nu(f) > 0$. Next by Proposition 2.15, $\nu(f)$ is the unique minimizer of $\Lambda^*_f$, and by strict convexity, $\Lambda^*_f$ is decreasing on $(-\infty, \nu(f)]$. Hence applying the upper LDP inequality (2.3.7) of Gärtner-Ellis Theorem 2.3.6 in [2], we get,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{V_t^{\pi^0} - V_0}{t} < \nu(f)/2 \right) \leq - \inf_{x \in (-\infty, \nu(f)/2]} \Lambda^*_f(x).$$
But

$$- \inf_{x \in (-\infty, \nu(f)/2]} \Lambda^*_f(x) = -\Lambda^*_f(\nu(f)/2).$$

These imply, under the defining $ALA$ hypothesis $V_0 = 0$, that

$$\mathbb{P}(V_{\pi_0}^t \geq \nu(f)t/2) \geq 1 - e^{-t\Lambda^*_f(\nu(f)/2)}$$

for all large time $t$.

To complete the proof of the theorem, it remains to check that $\Lambda^*_f(\nu(f)/2) > 0$, which clearly follows by Proposition 2.15 again since $\nu(f)/2 \neq \nu(f)$.

**Remark 2.17.** The additional research advance in the subject of asymptotic arbitrage theory provided by this new result is that the self-financing strategy $\pi_0^t$ generating $ALA$ with $GDP-F$ is explicitly constructed unlike in other works, as in [5], treating existence of earlier forms of asymptotic arbitrage.

**Example 2.18. (The Discrete-Time Ornstein-Uhlenbeck Process)** Consider the discrete-time Ornstein-Uhlenbeck (O-U) process,

$$S_{t+1} = \alpha S_t + \varepsilon_{t+1}, \text{ for all time } t \geq 1,$$

where $0 < |\alpha| < 1$ and $S_0$ are constants and $\varepsilon_t$ are $i.i.d \mathcal{N}(0,1)$. $S_t$ is also known as a stable auto-regressive process $AR(1)$.

In this stock prices model, the drift and volatility functions are identified as $\mu(x) = (\alpha - 1)x$ and $\sigma(x) = 1$, for all $x \in \mathbb{R}$, and are clearly measurable. Hence the market price of risk function is $\varphi(x) = (\alpha - 1)x$, for all $x \in \mathbb{R}$. The mean-reverting condition in (6) and all the remaining conditions of Theorem 2.4 trivially hold. From (8) we find $R_0 = \mathbb{R} \setminus \{0\} \equiv \mathbb{R}^\ast$, $R_0^+ = \mathbb{R}^+_\ast$ and $R_0^- = \mathbb{R}^-$. Obviously, $\lambda(R_0) = \infty > 0$. It follows that the corresponding constructed strategy $\pi_0^t = 1_{\mathbb{R}^+_\ast}(S_{t-1}) - 1_{\mathbb{R}^-}(S_{t-1})$ produces $ALA$ with $GDP-F$ in the investor's wealth (2) for this discrete-time O-U model of stock prices.

**Example 2.19. (A Cox-Ingersoll-Ross Type Process)** In Mathematical Finance the process described by the stochastic differential equation

$$dZ_t = -\beta Z_t dt + \sigma \sqrt{|Z_t|} dW_t$$

is often is called the Cox-Ingersoll-Ross (CIR) process and is used to model stochastic volatility or the short rate in bond markets. Here $W_t$ is Brownian
motion. We present a slight modification of the discretization of this model here. The modifications are necessary, since the volatility of \( Z_t \) is neither bounded above nor below. For that, let us define the stock prices process by

\[
S_{t+1} = \alpha S_t + \sigma \min\{\max\{\sqrt{|S_t|}, M\}, \eta\} \varepsilon_t, \quad t \geq 1,
\]

where \(|\alpha| < 1\), \( \sigma > 0\), \( 0 < \eta < M \) are given constants and \( \varepsilon_t \) are as in the preceding example.

It is easy to check that the CIR type process \( S_t \) also satisfies the conditions of Theorem 2.4.

### 3. Utility-Based Asymptotic Linear Arbitrage

For this section, the stock prices process \( S_t \), predictable (self-financing) strategies \( \pi_t \) and the corresponding wealth process \( V^\pi_t \) are still assumed relative to the same models and the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) of the introductory Section 1.

Consider the concept of expected utility of investors’ wealth discussed for e.g. in [4, Chap. 5]. If \( U : \mathbb{R} \to \mathbb{R} \) is a strictly increasing function, then \( U(V^\pi_t) \) represents the measure/level of satisfaction at time \( t \) for an investor using strategy \( \pi_t \) on his/her amount of wealth \( V^\pi_t \) with respect to the risk of losses. And \( \mathbb{E}U(V^\pi_t) \) is the expected level of such a satisfaction. \( U \) is called a utility function and it is strictly increasing because investors usually prefer more money than less. Moreover, utility functions are assumed either concave for risk-averse investors, or convex for risk-seeking investors, or linear for risk-neutral investors in the market. Usually, it is rare to see investors behaving in a risk-neutral way i.e., being indifferent between preferring a random (risky) outcome on their investments and a certain (riskless) amount of wealth. Hence, as announced at the end of the introduction, we introduce below the concept of utility-based asymptotic linear arbitrage only for risk-averse or risk-seeking investors.

**Definition 3.1.** Let \( U \) be any utility function (convex or concave). We say that a trading strategy \( \pi_t \) generates a utility-based asymptotic linear arbitrage with respect to \( U \) (abbreviated by \( U-ALA \) w.r.t. \( U \)), if starting from zero initial capital \( V_0 \) corresponding to zero or negative initial utility level \( U(V_0) \), the expected utility \( \mathbb{E}U(V^\pi_t) \) increases (at least) linearly fast in long-term, i.e., \( \mathbb{E}U(V^\pi_t) \geq b + ct \) for all large enough time \( t \geq 1 \), for some constants \( b \) and \( c > 0 \).
For the result of the section stated below, we consider only risk-seeking investors with the class of convex utility functions $U_{\alpha} : \mathbb{R} \to \mathbb{R}$, for a fixed real constant $\alpha > 0$, defined by $U_{\alpha}(x) := e^{\alpha x} - 1$, for all $x \in \mathbb{R}$. Similarly to the coefficient of constant risk-aversion (CARA) defined in [4, Chap. 5], $\alpha$ represents here the level of risk-seeking for those investors: the higher $\alpha$ is, the more an investor takes risk and may hence get higher satisfaction. This is typical to investors known as speculators in financial markets. Then we have the following

**Theorem 3.2.** Let $\pi_t$ be any trading strategy in the models (1) and (2). If $\pi_t$ is an ALA (with GDP-F), then $\pi_t$ also generates $U$-ALA w.r.t $U_{\alpha}$.

**Proof.** First, to the investor’s initial capital $V_0 = 0$, it corresponds the initial utility $U_{\alpha}(V_0) = e^0 - 1 = 0$. Next, by definition of ALA, there are a constant $a > 0$ and a time $t_{1/2}$ such that we have $P(V_{\pi t} \geq at) \geq 1/2$ for all time $t \geq t_{1/2}$. It follows by monotonicity and convexity of $U_{\alpha}$ that

$$
\mathbb{E}U_{\alpha}(V_{\pi t}) \geq -1 + \mathbb{E}U_{\alpha}(at)\mathbf{1}_{\{V_{\pi t} \geq at\}}
= -1 + U_{\alpha}(at)P(V_{\pi t} \geq at)
\geq -1 + (1/2)(e^{\alpha at} - 1)
\geq \frac{1}{2}(-3 + \alpha at),
$$

for all large enough time $t$, as required.

To conclude this section, let us explain how in practice this easily proved result may connect long-term arbitrageurs’ investment performances with market speculators’ level of satisfaction. Indeed, if a speculator investor risks higher by investing an amount of money on the stock $S_t$ and chooses a utility function $U_{\alpha}$, moreover if, as guaranteed by the existence Theorem 2.4, s/he manages to construct an ALA with $GDP$-$F$ strategy in the market models (1) and (2), then while his/her wealth grows linearly fast (with probability tending to 1), his/her expected level of satisfaction increases also (at least) linearly fast in long-term.

**References**


