OSCILLATION AND NON OSCILLATION
FOR THE SOLUTIONS OF CERTAIN TYPE OF
GENERALIZED NEUTRAL $\alpha$-DIFFERENCE EQUATION

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Abstract: In this paper, the authors discuss the oscillation and non oscillation for generalized neutral $\alpha$-difference equation

$$\Delta_{\alpha(\ell)} (u(k) + pu(k - \tau \ell)) + q(k)u(k - \sigma \ell) = 0, \quad k \in [0, \infty),$$

where $p$ is a constant, $q(k)$ is defined on $[0, \infty)$, $\tau$ is a positive integer and $\sigma$ is a non-negative integer.

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1. Introduction

The theory of difference equations is based on the operator $\Delta$ defined as

$$\Delta u(k) = u(k + 1) - u(k), \quad k \in \mathbb{N} = \{0, 1, 2, \ldots\}. \quad (2)$$

Even though many authors ([1],[9]) have suggested the definition of $\Delta$ as

$$\Delta u(k) = u(k + \ell) - u(k), \quad k \in [0, \infty), \quad \ell \in (0, \infty), \quad (3)$$

no significant progress took place on this line. But recently M. Maria Susai Manuel, G.B.A. Xavier and E. Thandapani [3], took up the definition of $\Delta$ as given in (3), and developed the theory of difference equations in a different direction and many interesting results were obtained in number theory. For convenience, the authors labeled the operator $\Delta$ defined by (3) as $\Delta_{\ell}$ and its inverse by $\Delta_{\ell}^{-1}$. When $\Delta_{\ell}$ is operated on a complex function $u(k)$ and considering $\ell$ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were noticed. The results obtained can be found in [3]-[7].

Jerzy Popenda [2], while discussing the behavior of solutions of a particular type of difference equation, defined $\Delta_{\alpha}$ as $\Delta_{\alpha} u(k) = u(k + 1) - \alpha u(k)$. This definition of $\Delta_{\alpha}$ is being ignored for a long time. Recently, M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier [8] have generalized the definition of $\Delta_{\alpha}$ by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)} u(k) = u(k + \ell) - \alpha u(k)$ for the real valued function $u(k)$ and $\ell \in (0, \infty)$ and also obtained the solutions of certain types of generalized $\alpha$-difference equations, in particular, the generalized Clairaut’s $\alpha$-difference equation, generalized Euler $\alpha$-difference equation and the generalized $\alpha$-Bernoulli polynomial $B_{\alpha(n)}(k, \ell)$, which is a solution of the $\alpha$-difference equation $u(k + \ell) - \alpha u(k) = nk^{n-1}$, for $n \in \mathbb{N}(1)$. In this paper, we present solutions of certain type of generalized neutral $\alpha$-difference equations and discuss the oscillatory and non oscillatory behavior of generalized neutral $\alpha$-difference equation (1).

Throughout this paper, we make use of the following assumptions:

$$\mu = \max\{\tau, \sigma\}.$$

Then by a solution of (1) we mean a function $u(k)$ which is defined for $k \geq -\mu$ and satisfies the equation (1) for $k \in [0, \infty)$. Clearly if

$$u(k) = A_k, \quad k \in [-\mu, 0] \quad (4)$$

are given, then (1) has a unique solution, and it can be constructed recursively. Also, assume that function $q(k)$ is not identically zero.
2. Preliminaries

In this section, we present the definition of the generalized $\alpha$-difference equation of the $n^{th}$ kind, from which the equation (1) becomes the generalized linear $\alpha$-difference equation of the third kind by properly selecting the values of $\ell_i$ for $i = 1, 2, 3$.

**Definition 2.1.** Let $L = \{\ell_1, \ell_2, \cdots, \ell_n\}$ be a set of $n$ positive real numbers, $r(L)$ be the set of all subsets of size $r$ from the set $L$ and $\alpha > 0$ be fixed. Then, for $k \in [0, \infty)$, we define the generalized $n^{th}$ kind $\alpha$-difference equation as

$$F \left( \left( k, (P_A(k, \alpha)u(k + \sum_{\ell_i \in A} a_i^A\ell_i))_{A \in r(L)} \right)^n \right)_{r=0} = 0, \quad (5)$$

and the generalized $n^{th}$ kind linear $\alpha$-difference equation as

$$\sum_{r=0}^n \sum_{A \in r(L)} P_A(k, \alpha)u(k + \sum_{\ell_i \in A} a_i^A\ell_i) = f(k), \quad (6)$$

where $P_A(k, \alpha)$, $f(k)$ and $F$ are real valued functions and $a_i^A$'s are constants.

**Remark 2.2.**

i) When $\ell_i = \ell$, for $i = 1, 2, \cdots, n$, the equation (5) (the equation (6)) becomes the generalized $n^{th}$ order (linear) $\alpha$-difference equation.

ii) When $\ell_i = 1$, for $i = 1, 2, \cdots, n$ and $k \in \mathbb{N}(a)$, $a$ is an integer, the equation (5) (the equation (6)) becomes the $n^{th}$ order (linear) $\alpha$-difference equation.

iii) When $\ell_i = 1$, for $i = 1, 2, \cdots, n$, $\alpha = 1$ and $k \in \mathbb{N}(a)$, $a$ is an integer, the equation (5) (the equation (6)) becomes the $n^{th}$ order (linear) difference equation.

iv) Equation (5) (the equation (6)) becomes the Delay or Neutral type difference equation by taking $\ell_i = 1$, for $i = 1, 2, \cdots, n$, $\alpha = 1$, $k \in \mathbb{N}(a)$, $a$ is an integer, negative values for certain $a_i$'s.

The following example illustrates Equation (6).

**Example 2.3.** Equation (1) can be expressed as

$$-\alpha u(k) + u(k + \ell) - p\alpha u(k - \tau\ell) + q(k)u(k - \sigma\ell) + pu(k + \ell - \tau\ell) = 0.$$  

By taking $\ell_1 = \ell$, $\ell_2 = \tau\ell$ and $\ell_3 = \sigma\ell$ we get $L = \{\ell, \tau\ell, \sigma\ell\},$
\begin{align*}
0(L) & = \{ \phi \}, \quad 1(L) = \{ \{ \ell \} \}, \quad 1(L) = \{ \{ \tau \ell \} \}, \\
2(L) & = \{ \{ \ell, \tau \ell \} \}, \quad \{ \ell, \sigma \ell \}, \quad \{ \tau \ell, \sigma \ell \} \} \quad \text{and} \quad 3(L) = \{ \{ \ell, \tau \ell, \sigma \ell \} \}.
\end{align*}

Now, if we take
\begin{align*}
P_{\{ \phi \}}(k, \alpha) & = -\alpha, \quad P_{\{ \ell \}}(k, \alpha) = 1, \quad P_{\{ \tau \ell \}}(k, \alpha) = -p\alpha, \quad P_{\{ \sigma \ell \}} = q(k), \\
P_{\{ \ell, \tau \ell \}}(k, \alpha) & = p, \quad P_{\{ \ell, \sigma \ell \}}(k, \alpha) = P_{\{ \tau \ell, \sigma \ell \}}(k, \alpha) = 0, \quad P_{\{ \ell, \tau \ell, \sigma \ell \}}(k, \alpha) = 0, \\
a_1^{\{ \ell \}} & = 1, \quad a_2^{\{ \tau \ell \}} = -1, \quad a_3^{\{ \sigma \ell \}} = -1, \quad a_1^{\{ \ell, \tau \ell \}} = -1, \quad a_2^{\{ \ell, \sigma \ell \}} = 1 \quad \text{and} \quad \text{all other } a_i^A \text{‘s are zero in } (6) \text{ then, equation (1) is a generalized third kind linear } \alpha \text{-difference equation.}
\end{align*}

**Definition 2.4.** A nontrivial solution \( u(k) \) of (1) is said to be oscillatory, if for every \( k > 0 \in [0, \infty) \) there exists a \( k \geq K \) such that \( u(k)u(k + \ell) \leq 0. \) The equation (1) itself is called oscillatory if all its solutions are oscillatory. Otherwise, it is called nonoscillatory.

### 3. Main Results

**Lemma 3.1.** Let \( \ell, \alpha > 0 \) and \( \alpha \neq 1. \) If \( v(k) \) is a solution of the generalized first order linear \( \alpha \)-difference equation
\begin{align*}
-\alpha v(k) + v(k + \ell) & = u(k), \quad (7)
\end{align*}

then \( w(k) = v(k) - \alpha^\left[ \frac{k}{\ell} \right] c_j, \quad (8) \)

where \( c_j \) is a constant for all \( k \in \mathbb{N}_\ell(j) \) is also a solution of (7).

**Proof.** Since (8) satisfies (7), the proof is obvious. \( \square \)

**Theorem 3.2.** Let \( u(k) \) be defined for all \( k \in [0, \infty) \). Then, for \( k \in [\ell, \infty), \)
\begin{align*}
v(k) = \sum_{r=1}^{\left[ \frac{k}{\ell} \right]} \alpha^{r-1} u(k - r\ell)
\end{align*}

is a solution of the generalized linear nonhomogeneous \( \alpha \)-difference equation
\begin{align*}
-\alpha v(k) + v(k + \ell) & = u(k). \quad (9)
\end{align*}

**Proof.** Replacing \( k \) by \( k - \ell \) and \( k - 2\ell \) in (9), we find
\begin{align*}
v(k) & = \alpha v(k - \ell) + u(k - \ell), \quad (10)
\end{align*}
and \( v(k - \ell) = \alpha v(k - 2\ell) + u(k - 2\ell), \) (11)
which yield \( v(k) = u(k - \ell) + \alpha u(k - 2\ell) + \alpha^2 v(k - 2\ell). \)
The proof follows by repeating this process again and again. \( \square \)

**Theorem 3.3.** Let \( \alpha \neq c^\ell, k \in [\ell, \infty) \) and \( j = k - \lfloor \frac{k}{\tau} \rfloor \ell. \) Then
\[
w(k) = \frac{kc^k}{(c^\ell - \alpha)} - \frac{\ell c^{k+\ell}}{(c^\ell - \alpha)^2} - \alpha \lfloor \frac{k}{\tau} \rfloor c_j, \tag{12}
\]
where \( c_j \) is a constant for all \( k \in \mathbb{N}_\ell(j) \) is a solution of the generalized first order linear \( \alpha \)-difference equation
\[
-\alpha v(k) + v(k + \ell) = kc^k. \tag{13}
\]

**Proof.** The proof follows by taking \( u(k) = kc^k \) in (7) and
\[
v(k) = \frac{kc^k}{(c^\ell - \alpha)} - \frac{\ell c^{k+\ell}}{(c^\ell - \alpha)^2}
\]
in (8). \( \square \)

**Lemma 3.4.** Let \( q(k) \geq 0 \) for all \( k \in [0, \infty) \) and let \( u(k) \) be an eventually positive solution of (1). Set \( z(k) = u(k) + pu(k - \tau\ell). \)

(a) If \( p = -1 \), then \( z(k) > 0 \) and \( \Delta_{\alpha(\ell)} z(k) \leq 0 \) eventually.
(b) If \( -1 < p < 0 \), then \( z(k) > 0 \) and \( \Delta_{\alpha(\ell)} z(k) < 0 \) eventually.
(c) If \( p < -1 \) and \( \sum_{k=1}^{\infty} p(k\ell + j) = \infty \), then \( z(k) < 0 \) and \( \Delta_{\alpha(\ell)} z(k) \leq 0 \) eventually.

**Proof.** Since \( q(k) \neq 0 \), from the equation (1), we have
\[
\Delta_{\alpha(\ell)} z(k) = -q(k) u(k - \sigma\ell) \leq 0,
\]
eventually, so \( z(k) \) cannot be eventually identically zero. Thus, it follows that \( z(k) \) is eventually positive or eventually negative.
If \( z(k) < 0 \) eventually, then \( z(k) \leq z(K) < 0 \) for \( k \geq K \in [0, \infty) \). Hence
\[
u(K + \tau k) \leq z(K) + u(K + (k - \ell)\tau) \leq \ldots \leq kz(K) + u(K).
\]
On letting $k \to \infty$ in the above inequality, we find $u(K + \tau k)$ to be negative, which is a contradiction to $u(k) > 0$. This proves (a).

The proof of (b) is similar to that of (a).

To prove (c), again from (1), we have $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k - \sigma \ell) \leq 0$, for all large $k$. We shall prove that $z(k) < 0$, eventually. If not, then $z(k) = u(k) + pu(k - \tau \ell) \geq 0, \quad k \geq K, 
\text{i.e.} \quad u(k) \geq -pu(k - \tau \ell), \quad k \geq K$

which implies that

$$0 < u(K - \tau \ell) \leq \left(\frac{-1}{p}\right) u(K) \leq \ldots \leq \left(\frac{-1}{p}\right)^r u(K + (r - 1)\tau \ell),$$

$r = 1, 2, \ldots$. On letting $r \to \infty$ in the above inequality, we get $u(k) \to \infty$ as $k \to \infty$. But, then since $\Delta_{\alpha(\ell)} z(k) = -q(k)u(k - \tau \ell) \leq -Lq(k)$ for large $k$, where $L$ is a positive number. On summing the last inequality, we obtain

$$z(k + \ell) - \alpha z(k) \leq -L \sum_{r=1}^{[\ell/p]} \alpha^{r-1} q(k - r\ell),$$

which implies that $z(k) \to -\infty$ as $k \to \infty$. This contradicts the fact that $z(k) \geq 0$ for $k \geq K$. \hfill \Box

Now we shall establish oscillation criteria for the difference equation (1). The results obtained depend on the values of the parameter $p$.

**Theorem 3.5.** Assume that $p = -1$, $q(k) \geq 0$ for $k \in \mathbb{N}(1)$, and for a positive integer $K$,

$$\sum_{r=0}^{\infty} \left(\frac{1}{\alpha}\right)^r q(K + r\ell) = \infty. \quad (14)$$

Then the equation (1) is oscillatory.

**Proof.** Assume the contrary. Without loss of generality let $u(k)$ be an eventually positive solution of (1). By Lemma (3.4), $z(k) = u(k) + pu(k - \tau \ell) > 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$, eventually. This implies that $\lim_{k \to \infty} z(k) = \gamma \geq 0$ exists.

On summing (1) from $K$ to $k$, we get

$$z(k + \ell) - \alpha z(k) + (1 - \alpha) \sum_{r=0}^{n-1} z(K + r\ell) + \sum_{r=0}^{n-1} \left(\frac{1}{\alpha}\right)^r q(r)u(K + r\ell - \sigma) = 0.$$
On letting $k \to \infty$ in the above equation, we obtain
\[ z(K) \geq \sum_{r=0}^{\infty} \left( \frac{1}{\alpha} \right)^r q(r) u(K + r\ell - \sigma). \] (15)

Now setting $\min_{K < r < K + \tau \ell} u(K + r\ell - \tau \ell) = s > 0$, we find from $z(k) = u(k) - u(k - \tau \ell) > 0$ for $k \geq K$ that $u(k) \geq s$ for $k \geq K$. Thus, from (15), we have
\[ \infty > z(K + r\ell + \sigma \ell) \geq \sum_{r=0}^{\infty} \left( \frac{1}{\alpha} \right)^r q(k + r\ell + \sigma \ell) u(k + r\ell - \sigma \ell) \]
\[ \geq \sum_{r=0}^{\infty} \left( \frac{1}{\alpha} \right)^r q(k + r\ell + \sigma \ell) \]
which contradicts condition (14).

**Example 3.6.** Consider the difference equation
\[ \Delta_{\alpha(\ell)}(u(k) - u(k - \tau \ell)) + 2(1 + \alpha)u(k - \sigma \ell) = 0, \quad k \in [0, \infty), \] (16)
where $\tau$ and $\sigma$ are odd and even positive integers respectively. Equation (16) satisfies the assumptions of Theorem 3.5, and therefore the equation (16) is oscillatory. In fact, $(-\alpha)^{\left\lceil \frac{k}{\tau} \right\rceil + 1}$ is an oscillatory solution of (16).

**Theorem 3.7.** The conclusion of Theorem 3.5 holds even if (14) is replaced by
\[ \sum_{s=0}^{\infty} \left( \frac{1}{\alpha} \right)^s (K + s\ell) q(K + s\ell) \sum_{r=0}^{\infty} \left( \frac{1}{\alpha} \right)^r q(k + r\ell) = \infty. \] (17)

**Proof.** Since (14) implies that the equation (1) is oscillatory, it suffices to show that all conditions of (1) oscillate in the case that
\[ \sum_{r=0}^{\infty} \left( \frac{1}{\alpha} \right)^r q(K + r\ell) < \infty. \] (18)

Assume for the sake of contradiction, that (1) has an eventually positive solution $u(k)$. Then, by Lemma (3.4)(a), $z(k) = u(k) - u(k - \tau \ell) > 0$ and $\Delta_{\alpha(\ell)} z(k) \leq 0$ eventually. Thus, eventually $u(k) > u(k - \tau \ell)$, which implies that there exists a constant $L > 0$ and $K \in [0, \infty)$ sufficiently large such that
u(k - ℓμ) ≥ L, k ≥ K. Thus, from \(\Delta_\alpha(k)z(k) = -q(k)u(k - σℓ)\) it follows that \(\Delta_\alpha(k)z(k) \leq -Lq(k), k ≥ K\) and hence \(z(k) ≥ L \sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ), k ≥ K\), which is the same as

\[
u(k) ≥ α(u(k - τℓ) + L \sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ), k ≥ K. \tag{19}\]

Now let \(I(k)\) denote the integer part of \(\frac{k-K}{τ}\), then we have

\[
u(k) ≥ L \left(\sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ) + \sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ - τℓ) + \cdots \right.
\]

\[+ \sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ - (I(k) - 1)τℓ)\] \[+ u(k - I(k)τℓ), \]

which together with \(\Delta_\alpha(k)z(k) = -q(k)u(k - σℓ)\) yields

\[
\Delta_\alpha(k)z(k) ≤ H(k), \tag{20}
\]

where \(H(k) = I(k)Lq(k) \sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ)\).

By noting the fact that \(I(k)/k \to 1/τℓ\) as \(k \to ∞\), we have

\[
H(k) \left(\sum_{r=0}^{∞} \left(\frac{1}{\alpha}\right)^r q(k + rℓ)\right)^{-1} = \frac{I(k)L}{k} \to \frac{L}{τℓ} \text{ as } k \to ∞. \tag{21}\]

Thus (17) and (21) imply that \(\sum_{r=0}^{∞} \left(\frac{1}{α}\right)^r H(K + rℓ) = ∞\), which together with (20) leads to \(z(k) \to -∞\) as \(k \to ∞\). This contradicts the hypothesis that \(z(k)\) is eventually positive. \(\square\)

**Example 3.8.** For the generalized neutral α-difference equation

\[
\Delta_\alpha(k)(u(k) - αu(k - τℓ)) + k^{-ηℓ}u(k - σℓ) = 0, \quad η ∈ (1, 3/2] \tag{22}\]

condition (17) is satisfied. Therefore, by Theorem 3.7 the equation (22) is oscillatory. However, the condition (14) does not satisfy.
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