International Journal of Applied Mathematics

Volume 27 No. 2 2014, 147-154
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v27i2.4

SINGULAR VALUES AND FIXED POINTS OF FAMILY OF FUNCTION $ze^z/(e^z-1)$

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Abstract: The aim of this paper is to study the singular values and fixed points of one parameter family of generating function, $h_\lambda(z) = \lambda\frac{ze^z}{e^z-1}$, $\lambda \in \mathbb{R}\backslash\{0\}$, which arises from the generalized Bernoulli generating function, Apostol-Bernoulli generating function or Stirling generating function. It is found that the function $h_\lambda(z)$ has infinitely many singular values. Further, it is shown that all the critical values of $h_\lambda(z)$ are lying outside the open disk centered at origin and having radius λ. Moreover, the real fixed points of $h_\lambda(z)$ for $\lambda < 0$ and their nature are investigated. Finally, the results found here are compared with the dynamical properties of functions $\lambda \tan z$, $E_\lambda(z) = \lambda\frac{e^z-1}{z}$ and $f_\lambda(z) = \lambda\frac{z}{z+4}e^z$ for $\lambda > 0$.

AMS Subject Classification: 30D05, 37C25, 58K05
Key Words: fixed points, critical values, singular values

1. Introduction

In the dynamics or iteration of functions, the singular values and fixed points play vital role to explore the dynamical behavior of functions. The dynamics of families of entire functions $\lambda\frac{\sinh(z)}{z}$ and $\lambda\frac{e^z-1}{z}$, $\lambda > 0$ with infinitely many bounded singular values are studied, respectively, by Prasad [6], and Kapoor and Prasad [3]. Prasad and Nayak [7] also found the dynamics of certain class

Received: February 14, 2014 © 2014 Academic Publications

of critically bounded entire transcendental functions. Further, the dynamical properties of bounded-type entire functions is investigated in [8]. The dynamics of $\lambda\frac{\sinh^2(z)}{z^4}$ with infinitely many bounded singular values is found in [9] and the dynamics of certain transcendental meromorphic functions with unbounded singular values is discussed in [5]. The theory of dynamics or iteration of transcendental functions can be seen in [2, 4]. The dynamics of functions with infinitely many bounded or unbounded singular values is scarcely found in comparison to functions with finitely many singular values. To fill this gap, the present work describes some dynamical properties of the family of transcendental meromorphic functions having unbounded singular values.

The existence of the present work is motivated by the relationship of our function $\frac{ze^z}{e^z-1}$ with well known generating functions. If choose $\mu = 1$ and $t = 1$ in the generalized Bernoulli generating function $\left(\frac{z}{e^z-1}\right)^\mu e^{tz} = \sum_{m=0}^{\infty} B_m^\mu(t)\frac{z^m}{m!}$; pick $\lambda = 1$ and $t = 1$ in the Apostol-Bernoulli generating function $\frac{ze^{tz}}{\lambda e^z-1} = \sum_{m=0}^{\infty} B_m(t,\lambda)\frac{z^m}{m!}$ and takes $t = 0$ in the Stirling generating function $\left(\frac{z}{1-e^{-z}}\right)^{t+1} = \sum_{m=0}^{\infty} S_m(t)\frac{z^m}{m!}$, then we get the function $\frac{ze^z}{e^z-1}$.

We devote this work by considering one parameter family of function $\frac{ze^z}{e^z-1}$ which arises from the generalized Bernoulli generating function, Apostol-Bernoulli generating function and Stirling generating function. To study this family of functions, we consider

$$\mathcal{H} = \left\{ h_\lambda(z) = \lambda\frac{ze^z}{e^z-1} \text{ and } h_\lambda(0) = \lambda : \lambda \in \mathbb{R}\backslash\{0\}, z \in \hat{\mathbb{C}} \cup \{\infty\} \right\}.$$

The function $h_\lambda \in \mathcal{H}$ is a transcendental meromorphic function with infinitely many poles at $\{2\pi ki : k \in \mathbb{Z}\}$. Further, all poles are simple and all are lying on imaginary axis. Moreover, the function $h_\lambda(z)$ is neither even nor odd and not periodic.

These are some definitions which are needed in the sequel. A point z is said to be a fixed point of function $f(z)$ if $f(z) = z$. A fixed point z_0 is called an attracting, neutral (indifferent) or repelling if $|f'(z_0)| < 1$, $|f'(z_0)| = 1$ or $|f'(z_0)| > 1$ respectively. A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{C}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0,\infty) \to \hat{C}$ satisfying $\lim_{t\to\infty}\gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or functions of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type.

The organization of the present paper is as follows: In Section 2, it is found that the function $h_\lambda \in \mathcal{H}$ has infinitely many singular values. Further, it is shown that all the critical values of $h_\lambda(z)$ are lying outside the open disk centered at origin and having radius λ. The existence of real fixed points of $h_\lambda(z)$ and their nature are investigated in Section 3. Finally, the results found here are compared with the dynamical properties of functions $\lambda \tan z$, $E_\lambda(z) = \lambda\frac{e^z-1}{z}$ and $f_\lambda(z) = \lambda\frac{z}{z+4}e^z$ for $\lambda > 0$.

2. Unbounded Singular Values of $h_\lambda \in \mathcal{H}$

In the present section, it is found that the function $h_\lambda \in \mathcal{H}$ has infinitely many singular values. Further, all the critical values of $h_\lambda(z)$ lie in the right half plane outside the disk centered at origin and having radius λ .

The following theorem shows that the function $h_\lambda(z)$ has infinitely many singular values:

Theorem 2.1. *Let $h_\lambda \in \mathcal{H}, \lambda \in \mathbb{R}\backslash\{0\}$. Then, the function $h_\lambda(z)$ possesses infinitely many singular values.*

Proof. Since $h'_\lambda(z) = 0$ for critical points, it gives

$$\lambda\frac{(e^z-1)(z+1)e^z - ze^ze^z}{(e^z-1)^2} = 0,$$

$$e^z - z - 1 = 0. \tag{1}$$

The real and imaginary parts of equation (1) are

$$\frac{y}{\sin y} - e^{y\cot y - 1} = 0, \tag{2}$$

$$x = y\cot y - 1. \tag{3}$$

The graph of the function $\frac{y}{\sin y} - e^{y\cot y-1}$ is shown in Figure 1.

It is seen that, from Figure 1, the equation (2) has infinitely many solutions. So, we assume that there exists a sequence $\{y_n\}_{n=-\infty,n\neq 0}^{n=\infty}$ such that equation 2 is satisfied, where $y_{-n} \in [-(2n+\frac{1}{2})\pi, -(2n+\frac{1}{4})\pi)$ and $y_n \in [(2n+\frac{1}{4})\pi, (2n+\frac{1}{2})\pi)$, $n = 1, 2, 3, \ldots$. Now, from equation (3), $x_n = 1 - y_n \cot y_n$ for nonzero integer n. Therefore, It follows that $h'_\lambda(z_n) = 0$. Consequently, $z_n = x_n + iy_n$ are critical points for $h_\lambda(z)$. The critical values of $h_\lambda(z)$ are given by $h_\lambda(z_n)$ for nonzero integer n. It is observed that $h_\lambda(z_n)$ are different for distinct n. It shows that the function $h_\lambda(z)$ has infinitely many critical values.

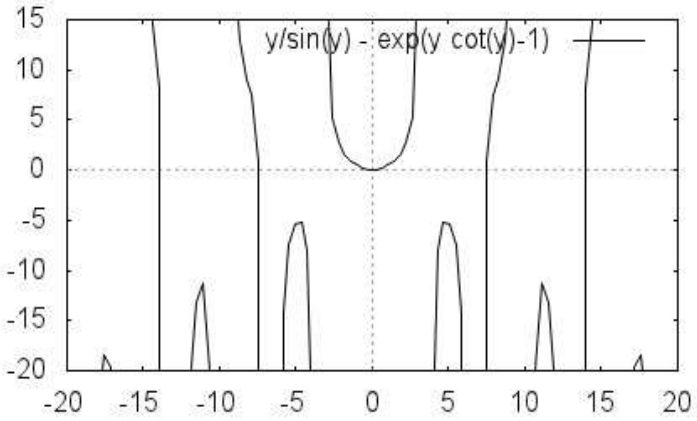


Figure 1: Graph of $\frac{y}{\sin y} - e^{y\cot y - 1}$

The finite asymptotic value of $h_\lambda(z)$ is 0 since $h_\lambda(z)$ tends 0 along negative real axis.

Thus, it follows that the function $h_\lambda(z)$ possesses infinitely many singular values. □

In the following theorem, it is found that the function $h'_\lambda(z)$ has no zeros in the left half plane:

Theorem 2.2. *Let $h_\lambda \in \mathcal{H}, \lambda \in \mathbb{R}\backslash\{0\}$. Then, the function $h'_\lambda(z)$ has no zeros in the left half plane $H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$.*

Proof. Since $h'_\lambda(z) = \lambda\frac{(e^z - z - 1)e^z}{(e^z-1)^2} = 0$, it gives $e^z = 1 + z$. Then,

$$e^x(\cos y + i \sin y) = 1 + x + iy. \tag{4}$$

When $y \neq 0$, then, by imaginary part of equation (4), $\frac{\sin y}{y} = e^{-x} > 1$ for $x < 0$. This is not true for both $y > 0$ and $y < 0$. When $y = 0$, then the real part of equation (4) gives $e^x = 1 + x$. For $x = -1$, it is trivially not hold. For $x < -1$, it is not true since $e^x > 0$ and $1 + x < 0$. It is also false for $-1 < x < 0$. Thus, the function $h'_\lambda(z)$ has no zeros in the left half plane H^-. □

The following theorem explores that the function $h_\lambda(z)$ maps the right half plane outside the open disk centered at origin and having radius λ.

Theorem 2.3. *Let $h_\lambda \in \mathcal{H}, \lambda \in \mathbb{R}\backslash\{0\}$. Then, the function $h_\lambda(z)$ maps the right half plane $H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}$ outside the open disk centered at origin and having radius λ.*

Proof. Suppose, the line segment γ is defined by $\gamma(t) = tz$, $t \in [0,1]$. Further, assume the function $g(z) = e^{-z}$ for an arbitrary fixed $z \in H^+$. Then

$$\int_\gamma g(z)dz = \int_0^1 g(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{-tz}dt = 1 - e^{-z}.$$

Since $M \equiv \max_{t\in[0,1]} |g(\gamma(t))| = \max_{t\in[0,1]} |e^{-tz}| < 1$ for $z \in H^+$. Now

$$|1 - e^{-z}| = \Big|\int_\gamma g(z)dz\Big| \leq M|z| < |z|,$$

so $|\frac{z}{1-e^{-z}}| > 1$ for all $z \in H^+$. Therefore, it follows that $|h_\lambda(z)| = |\lambda\frac{ze^z}{e^z-1}| > |\lambda|$ for all $z \in H^+$. This shows that the function $h_\lambda(z)$ maps the right half plane H^+ outside the open disk centered at origin and having radius λ. □

The following theorem proves that the function $h_\lambda \in \mathcal{H}$ has unbounded critical values:

Theorem 2.4. *Let $h_\lambda \in \mathcal{H}, \lambda \in \mathbb{R}\backslash\{0\}$. Then, the set of all the critical values of $h_\lambda(z)$ is unbounded subset of $\mathbb{C}\backslash\{$disk of radius $\lambda\}$.*

Proof. By Theorem 2.2, the function $h'_\lambda(z)$ has no zeros in the left half plane H^-. Consequently, all the critical points lie in the right half plane H^+. But, by using Theorem 2.3, the function $h_\lambda(z)$ maps the right half plane H^+ outside the open disk centered at origin and having radius λ. It means that all the critical values of $h_\lambda(z)$ are lying outside the open disk centered at origin and having radius λ. Hence, the set of all the critical values of $h_\lambda(z)$ is unbounded subset of $\mathbb{C}\backslash\{$disk of radius $\lambda\}$. □

3. Real Fixed Points of $h_\lambda \in \mathcal{H}$ and their Nature

In this section, the existence of real fixed points of the function $h_\lambda(x) = \lambda h(x)$, where $h(x) = \frac{xe^x}{e^x-1}$, and their nature are investigated. For sake of simplicity, consider the function $\phi(x) = xh'(x) + h(x)$. It gives that

$$\phi(x) = \frac{xe^x}{(e^x-1)^2}((2e^x - x - 2). \tag{5}$$

If $q(x) = 2e^x - x - 2$, then $q'(x) = 2e^x - 1$ and $q''(x) = 2e^x$. Since $q''(x) > 0$ for $x \in \mathbb{R}$, the function $q'(x)$ is increasing on $\mathbb{R}$. Moreover, $q'(0) = 1$ and

$q'(x) \to +\infty$ as $x \to +\infty$, it follows that there is a unique $\tilde{x} < 0$ such that $q'(x) < 0$ for $x < \tilde{x}$, $q'(\tilde{x}) = 0$ and $q'(x) > 0$ for $x > \tilde{x}$. Thus, $q(x)$ is decreases in $(-\infty, \tilde{x})$, attains its minimum at $\tilde{x}$ and increases afterward.

Since $q(0) = 0$ and $q(x) \to +\infty$ as $x \to +\infty$, it confirms that there is a unique negative $x^* < \tilde{x}$ such that $q(x) > 0$ for $x < x^*$, $q(x^*) = 0$ and $q(x) < 0$ for $x^* < x < 0$. Since $\frac{xe^x}{(e^x-1)^2} < 0$ for all $x < 0$. Now, we have

$$\phi(x) = \frac{xe^x}{(e^x-1)^2} q(x) \begin{cases} > 0 & \text{for } x^* < x < 0 \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x < x^* \end{cases} . \tag{6}$$

We define

$$\lambda^* = \frac{x^*}{h(x^*)} = 1 - e^{-x^*}, \tag{7}$$

where x^* is the unique negative real root of the equation $2e^x - x - 2 = 0$.

In the following theorem, it is shown that the function $h_\lambda(x)$ has a unique real fixed point:

Theorem 3.1. *Let $h_\lambda \in \mathcal{H}$ and $\lambda < 0$. Then, the function $h_\lambda(x)$ has a unique real fixed point x_λ.*

Proof. Since $h_\lambda(x) < 0$ for all $x \in \mathbb{R}$, each real fixed point of $h_\lambda(x)$ is negative. The function $h'_\lambda(x) = \lambda \frac{(e^x - x - 1)e^x}{(e^x-1)^2} < 0$ for $x \in \mathbb{R}$ and hence $h_\lambda(x)$ is decreasing on $\mathbb{R}$.

Let $\xi_\lambda(x) = h_\lambda(x) - x$ for $x \in \mathbb{R}$. Since $h'_\lambda(x) < 0$ for $x < 0$, it is observed that $\xi_\lambda(x)$ is decreasing. Since $\xi_\lambda(0) = \lambda < 0$, $\xi_\lambda(x) \to +\infty$ as $x \to -\infty$ and, due to continuity of $\xi_\lambda(x)$, there exists a unique negative $\hat{x} = x_\lambda$ such that $\xi_\lambda(\hat{x}) = 0$. Therefore, it gives that the function $h_\lambda(x)$, $\lambda < 0$, has a unique negative real fixed point x_λ. □

The nature of fixed points of $h_\lambda(x)$ are found in the following:

Theorem 3.2. *Let $h_\lambda \in \mathcal{H}$. Then, the fixed point x_λ of the function $h_\lambda(x)$ is (i) repelling for $\lambda < \lambda^*$ (ii) rationally indifferent for $\lambda = \lambda^*$ (iii) attracting for $\lambda^* < \lambda < 0$.*

Proof. Since the derivative of $\frac{x}{h(x)}$ is positive for $x < 0$, the function $\frac{x}{h(x)}$ is increasing on $\mathbb{R}^-$. We prove the following using this fact:

(i) For $\lambda < \lambda^*$, since the function $\frac{x}{h(x)}$ is increasing on $\mathbb{R}^-$ and $\lambda = \frac{x_\lambda}{h(x_\lambda)}$, we have $\frac{x_\lambda}{h(x_\lambda)} < \frac{x^*}{h(x^*)}$. It means that $-e^{-x_\lambda} < -e^{-x^*}$. Hence, $x_\lambda < x^*$. By Equation (6), $\phi(x_\lambda) < 0$. Since $h'_\lambda(x_\lambda) = \frac{\phi(x_\lambda)}{h(x_\lambda)} - 1$, it follows that

$h_\lambda(z) = \lambda\frac{ze^z}{e^z-1}$	$T_\lambda(z) = \lambda \tan z$	$E_\lambda(z) = \lambda\frac{e^z-1}{z}$	$f_\lambda(z) = \lambda\frac{z}{z+4}e^z$
$h_\lambda(z)$ is transcendental meromorphic function.	$T_\lambda(z)$ is transcendental meromorphic function.	$E_\lambda(z)$ is transcendental entire function.	$f_\lambda(z)$ is transcendental meromorphic function.
$h_\lambda(z)$ has infinitely many poles.	$T_\lambda(z)$ has infinitely many poles.	$E_\lambda(z)$ has no pole.	$f_\lambda(z)$ has only one pole.
$h_\lambda(z)$ has transcendental meromorphic Schwarzian Derivative.	$T_\lambda(z)$ has polynomial Schwarzian Derivative.	$E_\lambda(z)$ has transcendental meromorphic Schwarzian Derivative.	$f_\lambda(z)$ has rational Schwarzian Derivative.
$h_\lambda(z)$ is neither even nor odd function.	$T_\lambda(z)$ is an odd function.	$E_\lambda(z)$ is neither even nor odd function.	$f_\lambda(z)$ is neither even nor odd function.
$h_\lambda(z)$ has infinitely many critical values.	$T_\lambda(z)$ has no critical values.	$E_\lambda(z)$ has infinitely many critical values.	$f_\lambda(z)$ has finitely many critical values.
$h_\lambda(z)$ has only one finite asymptotic value 0.	$T_\lambda(z)$ has two finite asymptotic values $\pm\lambda i$.	$E_\lambda(z)$ has only one finite asymptotic value 0.	$f_\lambda(z)$ has only one finite asymptotic value 0.
All critical values of $h_\lambda(z)$ are unbounded.	All critical values of $T_\lambda(z)$ are bounded.	All critical values of $E_\lambda(z)$ are bounded.	All critical values of $f_\lambda(z)$ are bounded.

Table 1: Comparisons among the dynamical properties of the families of functions

$h'_\lambda(x_\lambda) + 1 = \frac{\phi(x_\lambda)}{h(x_\lambda)} < 0$. Since $h'_\lambda(x)$ is negative on $\mathbb{R}^-$, it shows that $|h'_\lambda(x_\lambda)| > 1$ and consequently, the fixed point x_λ of $h_\lambda(x)$ is a repelling for $\lambda < \lambda^*$.

(ii) For $\lambda = \lambda^*$, it is easy to prove $x_\lambda = x^*$. Now, by Equation (6), it follows that $h'_\lambda(x_\lambda)+1 = \frac{\phi(x_\lambda)}{h(x_\lambda)} = 0$ which implying $|h'_{\lambda^*}(x_\lambda)| = 1$. Therefore, the fixed point x^* of $h_\lambda(x)$ is rationally indifferent for $\lambda = \lambda^*$.

(iii) For $\lambda^* < \lambda < 0$, by similar arguments used in (i), it follows that $x_\lambda > x^*$. Again, by Equation (6) and by the fact $x_\lambda > x^*$, we have $\phi(x_\lambda) > 0$. It gives that $h'_\lambda(x_\lambda) + 1 = \frac{\phi(x_\lambda)}{h(x_\lambda)} > 0$ and hence $|h'_\lambda(x_\lambda)| < 1$. Therefore, x_λ is an attracting fixed point of $h(x_\lambda)$ for $\lambda^* < \lambda < 0$. □

It is observed that, from Theorem 3.2, the nature of the fixed point changes when parameter λ crosses parameter value λ^*.

Table 1 displays the comparisons among the dynamical properties of $h_\lambda(z) =$

$\lambda\frac{ze^z}{e^z-1}$ for $\lambda < 0$, $T_\lambda(z) = \lambda \tan z$ [1], $E_\lambda(z) = \lambda\frac{e^z-1}{z}$ [3] and $f_\lambda(z) = \lambda\frac{z}{z+4}e^z$ [10] for $\lambda > 0$.

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