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THE NON-COMPACTNESS EMBEDDINGS OF THE RADIAL SOBOLEV SPACES $\mathbf{H}^1_r(\mathbb{R}^n)$

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Abstract: The purpose of this paper is to give an elementary demonstration on the non-compactness embedding of radial Sobolev spaces $\mathrm{H}^1_r(\mathbb{R}^n)$ in $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^{2^*}(\mathbb{R}^n)$ where $2^* = \frac{2n}{n-2}$, $n > 2$. First we will show that the embedding $\mathrm{H}^1_r(\mathbb{R}^n) \to \mathrm{L}^q(\mathbb{R}^n)$ is compact for $2 < q < 2^*$, and then we give two examples for the critic cases $q = 2$ and $q = 2^*$ in which the compactness fails.

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1. Introduction and Basic Concepts

The Sobolev spaces are vital part of the language set for the theory of partial differential equations and for this reason it is very important to study their properties. In this paper we show a simple way non-compactness embedding of the radial Sobolev space $\mathrm{H}^1_r(\mathbb{R}^n)$ in $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^{2^*}(\mathbb{R}^n)$, where $2^* = \frac{2n}{n-2}$,

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$n > 2$.

One application on compactness embedding of radial Sobolev space is to obtain some existence theorems for nonlinear elliptic equations of the form

$$\begin{cases} -\Delta_p u + V(|x|)u^{p-1} = Q(|x|)u^{q-1}, u > 0, \text{ in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} |u(x)| = 0, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < n$, V and Q are continuous, nonnegative functions in $(0,\infty)$ (for more details, see [8]), or equations of the form

$$\begin{cases} -\Delta u + cu = a(x)|u|^{p-1} \text{ in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $c \geq 0$ is a constant and a is a given measurable function on $\mathbb{R}^n$, which is different from zero on a set of positive measures (see [1]).

Consider an open set $\Omega \subseteq \mathbb{R}^n$, $1 \leq p \leq \infty$ and let $m > 0$ be a non-negative integer. The *Sobolev space* $W^{m,p}(\Omega)$ is the collection of all functions in $\mathrm{L}^p(\Omega)$ such that all distribution derivatives up to order m are also in $\mathrm{L}^p(\Omega)$.

On the Sobolev space $W^{m,p}(\Omega)$ we shall use the norm

$$\|u\|_{m,p,\Omega} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{\mathrm{L}^p(\Omega)}. \tag{1}$$

Also, for $u \in W^{m,p}(\Omega)$ we will use the semi-norms

$$|u|_{m,p,\Omega} = \sum_{|\alpha|=m} \|D^\alpha u\|_{\mathrm{L}^p(\Omega)}, \tag{2}$$

and for $1 < p < \infty$,

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{\mathrm{L}^p(\Omega)} \right)^{1/p},$$

equivalent norm to (1).

In the special case where $p = 2$, the Sobolev space $W^{m,2}(\Omega)$ will be denoted by $\mathrm{H}^m(\Omega)$. For $u \in \mathrm{H}^m(\Omega)$, we denote its norm by $\|\cdot\|_{m,\Omega}$ and its semi-norm with $|\cdot|_{m,\Omega}$.

The space $\mathrm{H}^m(\Omega)$ is endowed with the inner product

$$(u,v)_{m,\Omega} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v, \qquad \text{for } u,v \in \mathrm{H}^m(\Omega).$$

Now, we define the radial Sobolev space $\mathrm{H}^1_r(\mathbb{R}^n)$ as the subset of the functions $u \in \mathrm{H}^1(\mathbb{R}^n)$ such that there is $f : [0,\infty) \to \mathbb{R}$ with

$$u(x) = f(|x|).$$

In 1977, Strauss showed that $\mathrm{H}^1_r(\mathbb{R}^n)$ is compact embedding in $\mathrm{L}^q(\mathbb{R}^n)$ for $2 < q < 2^*$, $n > 2$ (see Radial and Compactness Lemma in [7]). P-L. Lions in [6] also showed several results of the compact embedding of the radial Sobolev spaces. Similar results to those of Strauss and Lions, and of great importance were developed by Chabrowski [1]. For a proof the non-compact embedding, one can consult the full paper of Schonbek and Ebihara [3]. Although there are several versions of compact and non-compact embedding of radial Sobolev space $\mathrm{H}^1_r(\mathbb{R}^n)$, we present here an elementary proof of these facts.

2. Compact Embedding for $2 < q < 2^*$

This section shows that for $n > 2$, $\mathrm{H}^1_r(\mathbb{R}^n)$ is compact embedding in $\mathrm{L}^q(\mathbb{R}^n)$, $2 < q < 2^*$. It is worth noting that we use the standard notation $\mathcal{D}(\mathbb{R}^n)$ for the space of test functions and $\rightharpoonup$ indicates weak convergence.

Lemma 2.1. *Let $u \in \mathcal{D}(\mathbb{R}^n)$ a radial function. Then*

$$|u(x)| \le \Big(\frac{2}{\omega_n}\Big)^{\frac{1}{2}} |x|^{-(n-1)/2} |u|^{1/2}_{0,\mathbb{R}^n} |u|^{1/2}_{1,\mathbb{R}^n}, \tag{3}$$

where ω_n is the surface measure of the unit sphere in $\mathbb{R}^n$.

Proof. As $u \in \mathcal{D}(\mathbb{R}^n)$ is a radial function, there is $f : [0,\infty) \to \mathbb{R}$ such that $u(x) = f(r)$ with $r = |x|$. By the fundamental theorem of calculus,

$$r^{n-1}|f(r)|^2 = -\int_r^\infty \frac{d}{ds}(s^{n-1}|f(s)|^2)ds.$$

But,

$$(s^{n-1}|f(s)|^2)' = (n-1)s^{n-2}|f(s)|^2 + 2s^{n-1}|f(s)|f'(s).$$

So,

$$-\int_r^{\infty}(s^{n-1}|f(s)|^2)'ds. = -(n-1)\int_r^{\infty} s^{n-2}|f(s)|^2 ds - 2\int_r^{\infty} s^{n-1}|f(s)|f'(s)ds,$$

whereby

$$\begin{aligned} r^{n-1}|f(r)|^2 &\le 2\int_r^{\infty} s^{n-1}|f(s)||f'(s)|ds = 2\int_r^{\infty} |f(s)|s^{(n-1)/2}|f'(s)|s^{(n-1)/2}ds \\ &\le 2\left(\int_r^{\infty}|f(s)|^2 s^{n-1}\right)^{1/2}\left(\int_r^{\infty}|f'(s)|^2 s^{n-1}\right)^{1/2}. \end{aligned} \tag{4}$$

Since the function $u(x) = f(|x|)$, and using polar coordinates we obtain

$$|u|_{0,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n}|f(|x|)|^2 dx\right)^{1/2} = \omega_n^{1/2}\left(\int_0^{\infty}|f(s)|^2 s^{n-1}ds\right)^{1/2}. \tag{5}$$

Furthermore,

$$\frac{\partial u(x)}{\partial x_i} = f'(|x|)\frac{x_i}{|x|}$$

and so

$$\sum_{i=1}^{n}\left(\frac{\partial u(x)}{\partial x_i}\right)^2 = (f'(|x|))^2,$$

thus

$$|u|_{1,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n}|f'(|x|)|^2 dx\right)^{1/2} = \omega_n^{1/2}\left(\int_0^{\infty}|f'(s)|^2 s^{n-1}ds\right)^{1/2}. \tag{6}$$

Replaced (5) and (6) in (4), we obtain

$$r^{n-1}|f(r)|^2 \le \frac{2}{\omega_n}|u|_{0,\mathbb{R}^n}|u|_{1,\mathbb{R}^n},$$

i.e.,

$$|u(x)| \le (\frac{2}{\omega_n})^{1/2}|x|^{-(n-1)/2}|u|_{0,\mathbb{R}^n}^{1/2}|u|_{1,\mathbb{R}^n}^{1/2}.$$

□

Remark 1. If $u \in \mathrm{H}_r^1(\mathbb{R}^n)$, then (3) holds. Indeed, there is a sequence of functions $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^n)$ such that $u_k \to u$ in $\mathrm{H}_r^1(\mathbb{R}^n)$. Then

$$|u_k(x)| \le (\frac{2}{\omega_n})^{\frac{1}{2}}|x|^{-(n-1)/2}|u_k|_{0,\mathbb{R}^n}^{1/2}|u_k|_{1,\mathbb{R}^n}^{1/2},$$

taking the limit we obtain the result.

It is clear that, for $n > 2$, the inclusion $\mathrm{H}^1_r(\mathbb{R}^n) \subset \mathrm{L}^q(\mathbb{R})$ is continuous for $2 \leq q \leq \frac{2n}{n-2}$, it is a consequence of the Sobolev inequality and the interpolation of L^p spaces (see [4]).

Theorem 2.2. *Consider $n > 2$ and $2 < q < 2^*$, where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent. If u_k converges weakly to 0 in $\mathrm{H}^1_r(\mathbb{R}^n)$, then u_k converges strongly to 0 in $\mathrm{L}^q(\mathbb{R}^n)$.*

Proof. Given $R > 0$, we define $B^R = \{x : \|x\| > R\}$. By integrating over B^R in (3) we have

$$\int_{B^R} |u(x)|^q \leq C_n \int_{B^R} |x|^{\frac{-(n-1)q}{2}} = C_n \int_R^\infty \frac{1}{r^{(n-1)(\frac{q}{2}-1)}}.$$

The integral on the right side is convergent if $(n-1)(\frac{q}{2}-1) > 1$, i.e., if $q > \frac{2n}{n-1} > 2$ which we have by hypothesis. Then

$$\int_{B^R} |u(x)|^q \leq C_n R^{n(1-\frac{q}{2})+\frac{q}{2}},$$

doing have R to infinity, the right side of inequality tend to zero if $n(1-\frac{q}{2})+\frac{q}{2} < 0$, i.e., if $\frac{2n}{n-1} < q$. Now, by the Vitali convergence theorem the integral of the left side is convergent in $\mathrm{L}^r(\mathbb{R}^n)$ with $r = \frac{2n}{n-1}$. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{H}^1(\mathbb{R}^n)$ then it is bounded in $\mathrm{L}^{2^*}(\mathbb{R}^n)$. Using the hypothesis and interpolation of L_p spaces have

$$\|u_n\| \leq \|u_n\|_r^\lambda \|u_n\|_{2^*}^{1-\lambda}$$

from which we can conclude that $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathrm{L}^q(\mathbb{R}^n)$. □

3. Non-Compact Embedding for Cases $q = 2$ and $q = 2^*$

Now we study that for $n > 2$, $\mathrm{H}^1_r(\mathbb{R}^n)$ is not compactly embedded in $\mathrm{L}^q(\mathbb{R}^n)$ when $q = 2$ and $q = 2^*$.

a) Case $q = 2$

Consider $u \in \mathrm{H}^1_r(\mathbb{R}^n)$ such that $|u|_{0,\mathbb{R}^n} = 1$. For $k = 1, 2, \ldots$, we define

$$u_k(x) = k^{-\frac{n}{2}} u\left(\frac{x}{k}\right).$$

Note that

$$|u_k|^2_{0,\mathbb{R}^n} = k^{-n} \int_{\mathbb{R}^n} \left| u\left(\frac{x}{k}\right)\right|^2 dx = \int_{\mathbb{R}^n} |u(x)|^2\, dx = |u|^2_{0,\mathbb{R}^n} = 1,$$

and

$$\frac{\partial u_k(x)}{\partial x_i} = k^{-\frac{n}{2}} \frac{1}{k} \frac{\partial u(\frac{x}{k})}{\partial x_i}, \qquad i = 1, \dots, n,$$

and thereby

$$|u_k|_{1,\mathbb{R}^n} = \frac{1}{k}|u|_{1,\mathbb{R}^n} \to 0, \text{ when } k \to \infty.$$

Then it follows that $(u_k)_{k\mathbb{N}}$ is a bounded sequence in $\mathrm{H}^1_r(\mathbb{R}^n)$ that has a convergent subsequence in $\mathrm{L}^2(\mathbb{R}^n)$.

b) Case $q = 2^*$

We choose $u \in \mathrm{H}^1_r(\mathbb{R}^n)$ such that $|u|_{1,\mathbb{R}^n} = 1$. Define

$$u_k(x) = k^{(n-2)/2} u(kx).$$

Thus,

$$|u_k|_{0,\mathbb{R}^n} = \left(k^{n-2} \int_{\mathbb{R}^n} |u(kx)|^2\, dx\right)^{\frac{1}{2}} = \frac{1}{k}|u|_{0,\mathbb{R}^n} \to 0, \quad \text{as} \quad k \to \infty.$$

Moreover,

$$\frac{\partial u_k(x)}{\partial x_i} = k^{(n-2)/2} k \frac{\partial u(kx)}{\partial x_i}, \quad \text{for} \quad i = 1, \dots, n,$$

whereby,

$$|u_k|_{1,\mathbb{R}^n} = |u|_{1,\mathbb{R}^n} = 1.$$

Furthermore,

$$|u_k|_{0,2^*,\mathbb{R}^n} = |u|_{0,2^*,\mathbb{R}^n} \le C\|u\|_{1,\mathbb{R}^n},$$

is a sequence of functions in $\mathrm{L}^{2^*}(\mathbb{R}^n)$ that has a convergent subsequence.

References

[1] J. Chabrowski, On compact embeddings of radial Sobolev spaces and their applications, *Comm. in PDE*, **17**, No 5-6 (1992), 941–952.

[2] D.G. De Figueiredo, E.M. Dos Santos, and O.H. Miyagaki, Sobolev spaces of symmetric functions and applications, *J. Funct. Anal.*, **261**, No 12 (2011), 3735–3770.

[3] Y. Ebihara and T.P. Schonbek, On the (non)compactness of the radial Sobolev Spaces, *Hiroshima Math. J.*, **16** (1986), 665–669.

[4] S. Kesavan, *Topics in Functional Analysis and Applications*, John Wiley & Sons (1989).

[5] G. Leoni, *A First Course in Sobolev Spaces*, American Mathematical Society (1967).

[6] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, *J. Funct.Anal.*, **49** (1982), 315–334.

[7] W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, **55** (1977), 149–162.

[8] J. Su, Z.-Q. Wang, and M. Willem, Weighted Sobolev embedding with unbounded and decaying radial potentials, *J. Differential Equations*, **238** (2007), 201–219.