CAUCHY PROBLEMS WITH MODIFIED CONDITIONS FOR THE EULER-POISSON-DARBOUX EQUATIONS IN THE SPHERICAL SPACE

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Abstract: In this note we give the solutions of the Cauchy problems for the Euler-Poisson-Darboux equations (EPD) with modified conditions in the spherical space with application to the wave equation.

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1. Introduction

In El-Hafedh and Ould Moustapha [1] and El-Hafedh et al. [2], there are obtained the explicit solutions of Cauchy problems with modified conditions for the Euler-Poisson-Darboux equations in Euclidean and hyperbolic spaces. Here we give the explicit solutions of Cauchy problems with modified conditions for the Euler-Poisson-Darboux equation in spherical space. The classical Cauchy problem for the Euler-Poisson-Darboux equation in spherical space is considered in Fusaro [3] and in Kipriyanov and Ivanov [4] and [5]:

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More specifically, we are interested in the family of problems:

\begin{align*}
\tag{E_{\mu}^n}''
(a) \quad L_n U(t, \theta) &= A_{t}^\mu U(t, \theta) \quad 0 < t < \pi, \ \theta \in S^n \\
(b)'' \quad U(0, \theta) &= f(\theta), \ \partial_t U(0, \theta) = 0; \ f \in C^\infty(S^n).
\end{align*}

The operator $A_{t}^\mu$ is given by:

\begin{equation}
A_{t}^\mu = \frac{\partial^2}{\partial t^2} + (1 - 2\nu) \cot \theta \frac{\partial}{\partial \theta} - \left(1 - \frac{2\nu}{2}\right)^2.
\end{equation}

Note that in $(E_{\mu}^n)''$, the second data is zero ($g = 0$) as a solution of equation (a) can not be regular for $t = 0$ if its first derivative with respect to $t$ are not zero. The modified conditions (b) and (b) allow to take the second data as any function $g$, void or while covering the classical Cauchy conditions (b)'. Thus the Cauchy problems $(E_{\frac{1}{2}}^n)$ and $(E_{\frac{1}{2}}')$ correspond to the classical (see Bunke and Olbrich[7]) and radial (Theorem 2) wave equations in $S^n$.

The main results of this note - Theorems 1, 2 and 3 - are given below, and their applications are in Section 6.

\textbf{2. Theorems}

\textbf{Theorem 1.} (Classical EPD with modified initial conditions) Let $\mu \in (0, \frac{1}{2})$. The Cauchy problem $(E_{\mu}^n)$ with modified conditions for the classical Euler-Poisson-Darboux equation on the spherical space has the unique solution given
by:

$$U(t, \theta) = \alpha_{n,-\mu}(\sin t)^{2\mu} \left( \frac{\partial}{\sin t \partial t} \right)^{\frac{n-1}{2}} \int_{r<t} f(\theta') \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{-\mu - \frac{1}{2}} d\mu(\theta')$$

$$+ \frac{1}{2\mu} \alpha_{n,\mu} \left( \frac{\partial}{\sin t \partial t} \right)^{\frac{n-1}{2}} \int_{r<t} g(\theta') \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{\mu - \frac{1}{2}} d\mu(\theta'),$$

when $n$ is odd;

$$U(t, \theta) = \beta_{n,-\mu}(\sin t)^{2\mu} \left( \frac{\partial}{\sin t \partial t} \right)^{\frac{n}{2}} \int_{r<t} f(\theta') \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{-\mu} d\mu(\theta')$$

$$+ \frac{1}{2\mu} \beta_{n,\mu} \left( \frac{\partial}{\sin t \partial t} \right)^{\frac{n}{2}} \int_{r<t} g(\theta') \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{\mu} d\mu(\theta'),$$

when $n$ is even; where $\alpha_{n,\mu} = \frac{1}{2\mu} \frac{\Gamma(1 + 2\mu)}{(2\pi)^{\frac{n}{2}} \Gamma^{2}(\frac{1}{2} + \mu)}$, $\beta_{n,\mu} = \frac{4\mu}{(2\pi)^{\frac{n}{2}}}$ and $r = d(\theta, \theta')$ is the geodesic distance between $\theta$ and $\theta'$ in $\mathbb{S}^n$.

**Theorem 2.** (Radial wave equation) Let $1 - 2\nu = 2q$ be a positive integer. The Cauchy problem $\left( E^\mu_{\frac{\nu}{2}} \right)'$ for the radial wave equation on the spherical space has the unique solution given by:

$$U(t, \theta) = \int_0^\pi f(\theta') \frac{\partial}{\partial t} W(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta' + \int_0^\pi g(\theta') W(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta',$$

where

$$W(t, \theta, \theta') = 4^{\nu-1} i \left( \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \right)^{\nu} \left( \cos \frac{\theta'}{2} \right)^{2\nu}$$

$$\times \int_0^{+\infty} \int_{z e^{-\frac{i}{2}}}^{z e^{\frac{i}{2}}} J_{-\nu}(z \sin \frac{\theta}{2}) J_{-\nu}(z' \sin \frac{\theta'}{2})$$

and $J_\nu$ is the Bessel function (see [6], p. 65).

**Theorem 3.** (Radial EPD with modified initial conditions) Let $1 - 2\nu = 2q$ be a positive integer and $\mu \in (0, \frac{1}{2})$. The Cauchy problem $\left( E^\mu_{\frac{\nu}{2}} \right)'$ with modified conditions for the radial Euler-Poisson-Darboux equation on the spherical space has the unique solution given by:

$$U(t, \theta) = (\sin t)^{2\mu} \int_0^\pi f(\theta') W_{-\mu}(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta'$$
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\[ + \frac{1}{2\mu} \int_0^\pi g(\theta')W_\mu(t, \theta, \theta')(\sin \theta')^{1-2\nu} \, d\theta', \]

where

\[ W_\mu(t, \theta, \theta') = 4^{\nu+\mu-1}i \frac{\Gamma(1+\mu)}{\sqrt{\pi} \Gamma(\frac{1}{2}+\mu)} \left( \frac{\sin \frac{\theta}{2} \sin \frac{\theta'}{2}}{\cos \frac{\theta'}{2}} \right)^\nu \]

\[ \times \int_0^t \left( \sin^2 \frac{t}{2} - \sin^2 \frac{y}{2} \right)^{\nu-\frac{1}{2}} \frac{\partial}{\partial y} \int_0^{+\infty} \int_{z_0}^{\infty} J_{-\nu}(z \sin \frac{\theta}{2}) J_{-\nu}(z' \sin \frac{\theta'}{2}) \]

\[ \times J_0 \left( \sqrt{z^2 + z'^2 - 2zz' \cos \frac{y}{2}} \right) z^{-\nu} z'^\nu \, dz \, dz' \, dy. \quad (2.2) \]

### 3. Preliminaries

In this section we recall the continuous Jacobi transform, see Walter and Zayed [8], and we give some lemmas.

If \( 1-2\nu = 2q \) is a positive integer and \( f(x)(1+x)^\nu \in L^1 \{ (-1, 1), (1-x^2)^{-\nu} \} \), the continuous Jacobi transform \( \hat{f}(\lambda) \) of \( f(x) \) is defined by

\[ \hat{f}(\lambda) = \frac{1}{4q} \int_{-1}^1 f(x) P_\lambda^{\nu}(x) (1-x^2)^{-\nu} \, dx, \quad \lambda > \nu - \frac{1}{2}, \quad (3.1) \]

where \( P_\lambda^{\nu}(x) \) is the Jacobi function of the first kind, namely:

\[ P_\lambda^{\nu}(x) = \frac{\Gamma(\lambda + 1 - \nu)}{\Gamma(\lambda + 1) \Gamma(1-\nu)} F \left( -\lambda, \lambda + 1 - 2\nu, 1 - \nu, \frac{1-x}{2} \right), \quad (3.2) \]

and \( F(a, b, c, z) \) is the Gauss hypergeometric function, see [6].

**Lemma 1.** We have \( \overline{A_\theta^{\nu}} f(\lambda) = - (\lambda + q)^2 \hat{f}(\lambda) \), where \( \lambda \in \mathbb{R}^+, \quad q = \frac{1}{2} - \nu. \)

**Proof.** It suffices to write

\[ A_\theta^{\nu} = \frac{1}{(\sin \theta)^{1-2\nu}} \frac{\partial}{\partial \theta} (\sin \theta)^{1-2\nu} \frac{\partial}{\partial \theta} + \left( \frac{1-2\nu}{2} \right)^2, \]
Lemma 2. An inverse transform of the continuous Jacobi transform (3.1) is given by:

\[ f(x) = 4\pi \int_0^{+\infty} \frac{\Gamma(\lambda + q)}{\Gamma(\lambda + 1/2)} \left[ \frac{1}{\Gamma(\lambda + q)} P_{\lambda-q}^\nu(-x) \frac{\lambda \cot[(q-\lambda)\pi]}{\Gamma(q-\lambda)\Gamma(q+\lambda)} \right] d\lambda. \] (3.3)

Proof. By using the properties of the Gamma function (see Magnus et al. [6], p. 2), we unify several formulas for the inverse transform in Walter and Zayed [8] ((5.1),(5.4),(5.9) and (5.11)) when \(2q\) is a positive integer.

Lemma 3. For \(0 < t, \theta, \theta' < \pi\),

\[ J(t,\theta,\theta') = \int_0^{+\infty} \int_{ze^{-\frac{t}{2}}}^{ze^{\frac{t}{2}}} J_{-\nu}(z \sin \frac{\theta}{2}) J_{-\nu}(z' \sin \frac{\theta'}{2}) \times J_0 \left( \sqrt{z^2 + z'^2 - 2zz' \cos \frac{t}{2}} \right) z^{-\nu} z'^{-\nu} dz dz', \]

then if \(t\) is sufficiently small, we have the following asymptotic formula:

\[ J(t,\theta,\theta') \approx \frac{i \sin \frac{t}{2}}{2\pi \sin \frac{\theta}{2} \sin \frac{\theta'}{2}} \int_{-1}^{1} \frac{1}{\sqrt{Z}} F \left( \frac{1}{2} - \nu, \frac{1}{2} + \nu, \frac{1}{2}, Z \right) dp, \]

where \(Z = \frac{a^2 - (b-c)^2}{4bc}\), \(b - a < c < b + a\), \(a = \sqrt{1 - p^2 \sin \frac{t}{2}}\), \(b = \sin \frac{\theta}{2}\), \(c = \sin \frac{\theta'}{2}\).

Lemma 4. If \(W_{-\mu}^\nu\) is a solution of \((a)',\) then we have:

(i) \(A_t^\mu \left[ (\sin t)^{2\mu} W_{-\mu}^\nu(t,\theta) \right] = (\sin t)^{2\mu} A_t^{-\mu} W_{-\mu}^\nu(t,\theta);\)

(ii) \((\sin t)^{2\mu} W_{-\mu}^\nu(t,\theta)\) satisfies equation \((a)'\) in \((E_{-\mu}^\nu)';\)

(iii) \(W_{-\mu}^{1-\frac{n}{\pi}}(t, r)\) and \((\sin t)^{2\mu} W_{-\mu}^{1-\frac{n}{\pi}}(t, r)\) satisfies equation \((a)\)

with \(r = d(\theta, \theta')\).
Lemma 5. For $0 < t < \pi$ and $\theta, \theta' \in S^n$ let

$$W_{n,\mu}(t, \theta, \theta') = C_{n,\mu} \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{\mu - \frac{n}{2}}$$

with $C_{n,\mu} = \frac{4\mu \Gamma(1 + \mu)}{2^n \pi^\frac{n}{2} \Gamma(1 + \mu - \frac{n}{2})}$ and $r = d(\theta, \theta')$, then we have:

(i) $W_{n,\mu}(t, \theta, \theta') = \left\{ \begin{array}{ll}
\alpha_{n,\mu}(\frac{\partial}{\partial \sin \theta}) \frac{n-1}{2} \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{\mu - \frac{1}{2}} & \text{when } n \text{ is odd} \\
\beta_{n,\mu}(\frac{\partial}{\partial \sin \theta}) \frac{n}{2} \left( \sin^2 \frac{t}{2} - \sin^2 \frac{r}{2} \right)^{n} & \text{when } n \text{ is even},
\end{array} \right.$

(ii) $W_{n,\mu}(t, x, x')$ satisfies the equation (a).

Lemma 6. Let $J_\nu$ be the Bessel function, then we have:

(i) $A_\nu' \left[ \sin^\nu \frac{\theta}{2} J_{-\nu}(z \sin \frac{\theta}{2}) \right] = -\frac{1}{4} \sin^\nu \frac{\theta}{2} B_\nu' \left[ J_{-\nu}(z \sin \frac{\theta}{2}) \right]$,

(ii) $A_\nu'' \left[ \sin^\nu \frac{\theta}{2} \cos \nu \frac{\theta}{2} J_{-\nu}(z \sin \frac{\theta}{2}) \right] = -\frac{1}{4} \sin^\nu \frac{\theta}{2} \cos \nu \frac{\theta}{2} B_{-\nu}' \left[ J_{-\nu}(z \sin \frac{\theta}{2}) \right]$

where $B_\nu' = z^2 \frac{\partial^2}{\partial z^2} + (3 - 2\nu)z \frac{\partial}{\partial z} + (1 - \nu)^2 + z^2$.

(iii) $\int (B_\nu \phi) \psi dz = \int \phi (C_\nu \psi) dz$, for $\phi \in L^1_{\text{loc}}(\mathbb{R}^+) \text{ and } \psi \in D(\mathbb{R}^+)$

with $C_\nu = z^2 \frac{\partial^2}{\partial z^2} + (1 + 2\nu)z \frac{\partial}{\partial z} + \nu^2 + z^2$.

(iv) The function $\psi(t, z, z') = z^{-\nu} J_0 \left( \sqrt{z^2 + z'^2 - 2zz' \cos \frac{t}{2}} \right)$ satisfies the equation $-\frac{1}{4} C_\nu \psi(t, z, z') = \frac{\partial^2}{\partial t^2} \psi(t, z, z')$.

The proofs of Lemmas 3, 4, 5 and 6 are analogous of the corresponding lemmas in El-hafed et al. [2] and are left to the reader.

4. The Classical Euler-Poisson-Darboux Equation

Proof of Theorem 1. – To prove that $U(t, \theta)$ satisfies equation (a), we use Lemmas 4 and 5.

– To see the initial conditions, we introduce the polar coordinates centralized in $\theta$: $\theta' = \theta + \tan \frac{\omega}{2} \nu$, $\omega \in S^{n-1}$, and the change of variable $\sin \frac{\omega}{2} = (\sin \frac{t}{2}) s$, $0 < s < 1$, we obtain:

$$U(t, \theta) = 2^{n+2\mu} C_{n-\mu} \cos 2\mu \frac{t}{2} \int_0^1 f'_{\theta} \left( \frac{\psi}{2} \sin \frac{t}{2} \right)(1 - s^2)^{-\mu - \frac{\nu}{2}}$$

$$\times \left( 1 + s^2 \sin^2 \left( \frac{t}{2} \right) \right)^{\frac{n-2}{2}} s^{-1} ds + \frac{C_{n,\mu}}{2\mu} 2^n \sin^{2\mu} \frac{t}{2}$$
\[ \times \int_0^1 g^\#(\varphi \sin \frac{t}{2})(1 - s^2)^{\mu - \frac{n}{2}} \left(1 + s^2 \sin^2 \frac{t}{2}\right)^{\frac{n-2}{2}} s^{n-1} ds, \]

where \( f^\#(r) = \int_{S^{n-1}} f(\theta + r\omega) d\sigma(\omega) \) and \( \varphi = \frac{s}{\sqrt{1 + s^2 \sin^2 \frac{t}{2}}} \), since \( \int_{S^{n-1}} d\sigma(\omega) = \frac{2\pi}{\Gamma\left(\frac{n}{2}\right)} \) and \( \int_0^1 (1 - s^2)^{-\frac{\mu}{2}} s^{n-1} ds = \frac{\Gamma(1 - \mu - \frac{n}{2})\Gamma\left(\frac{n}{2}\right)}{2\Gamma(1 - \mu)} \).

5. The Radial Wave Equation

Proof of Theorem 2. – To prove that the kernel \( W(t, \theta, \theta') \) given in (2.1) satisfies the equation \((a)'\), \( (\mu = \frac{1}{2}) \), we use Lemma 6.

– To see the initial conditions, we use Lemma 3 and the change of variables:

\[ \theta' = 2 \arcsin \left( \frac{\sin \theta + q\sqrt{1 - p^2 \sin^2 \frac{t}{2}}}{\cos \frac{t}{2} + ip \sin \frac{t}{2}} \right), \quad -1 < p, q < 1. \]

Remark 1. By applying the Jacobi transform (3.1) to this problem, we have from Lemma 1:

\[ \hat{U}(t, \lambda - q) = \cos(\lambda t) \hat{f}(\lambda - q) + \frac{\sin(\lambda t)}{\lambda} \hat{g}(\lambda - q). \quad (5.1) \]

By using the inversion formula (3.3) and interchange the order of integration we have from Lemma 2:

\[ U(t, \theta) = \int_0^\pi f(\theta') \frac{\partial}{\partial t} W(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta' \]

\[ + \int_0^\pi g(\theta') W(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta', \]

\[ W(t, \theta, \theta') = \frac{\pi}{4^{\nu-1}} \int_0^\infty \left[ \frac{\Gamma(\lambda + q)}{\Gamma(\lambda + \frac{1}{2})} \right]^2 P_{\lambda - q}(\cos \theta)P_{\lambda - q}(\cos \theta') \]

\[ \times \frac{\sin(\lambda t) \cot[(q - \lambda)\pi]}{\Gamma(q - \lambda)\Gamma(q + \lambda)} d\lambda. \]
6. The Radial Euler-Poisson-Darboux Equation

Proof of Theorem 3. – To prove that $U(t, \theta)$ satisfies equation (a)', we use Lemmas 4, 5 and 6.
– To see the initial conditions, we use Lemma 3 and the change of variables:

$$\sin \frac{y}{2} = s \sin \frac{t}{2} \text{ and } \theta' = 2 \arcsin \left( \sin \frac{\theta}{2} + q \sqrt{1 - p^2 s \sin \frac{t}{2}} \right), \quad -1 < q < 1.$$ 

Remark 2. By applying the Jacobi transform to this problem, we have from Lemma 1:

$$\hat{U}(t, \lambda - q) = (\cos \frac{t}{2})^{2\mu} 2 F_1 \left( \frac{1}{2} - \lambda, \frac{1}{2} + \lambda, 1 - \mu, \sin^2 \frac{t}{2} \right) \hat{f}(\lambda - q)$$ 

$$+ \frac{1}{2\mu} (2\sin \frac{t}{2})^{2\mu} 2 F_1 \left( \frac{1}{2} - \lambda, \frac{1}{2} + \lambda, 1 + \mu, \sin^2 \frac{t}{2} \right) \hat{g}(\lambda - q). \quad (6.1)$$

By the inversion formula and interchanging the order of integration, we have from Lemma 2:

$$U(t, \theta) = (\sin t)^{2\mu} \int_0^\pi f(\theta') W_{-\mu}(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta'$$

$$+ \frac{1}{2\mu} \int_0^\pi g(\theta') W_{\mu}(t, \theta, \theta')(\sin \theta')^{1-2\nu} d\theta',$$

$$W_{\mu}(t, \theta, \theta') = \frac{\pi}{4q-1} \left( 2\sin \frac{t}{2} \right)^{2\mu} \int_0^{+\infty} \left[ \frac{\Gamma(\lambda + q)}{\Gamma(\lambda + \frac{1}{2})} \right]^2 P_{\lambda-q}^\nu(\cos \theta) P_{\lambda-q}^\nu(\cos \theta')$$

$$\times 2 F_1 \left( \frac{1}{2} - \lambda, \frac{1}{2} + \lambda, 1 + \mu, \sin^2 \frac{t}{2} \right) \frac{\lambda \cot[(q - \lambda)\pi]}{\Gamma(q - \lambda)\Gamma(q + \lambda)} d\lambda.$$ 

7. Applications

Corollary 1. (Bunke and Olbrich [7], Proposition 2.2) The classical wave equation in the spherical space of dimension $n$). We let $\mu \to \frac{1}{2}$ in Theorem 1, we obtain the solution of the Cauchy problem for the classical wave equation in $\mathbb{S}^n$ ($f = 0$):

$$U(t, \theta) = \frac{1}{2(2\pi)^m} \left( \frac{\partial}{\sin t \partial t} \right)^{m-1} \frac{1}{\sin t} \int_{S_t(\theta)} g(\theta') d\mu(\theta'),$$
when \( n \) is odd \( (n = 2m + 1) \), where \( S_t(\theta) \) is the sphere of radius \( t \) around \( \theta \);

\[
U(t, \theta) = \frac{1}{\sqrt{2}(2\pi)^m} \left( \frac{\partial}{\sin t \partial t} \right)^{m-1} \int_{S^n} g(\theta') \Re \frac{1}{\sqrt{\cos(t) - \cos(d(\theta, \theta'))}} d\mu(\theta'),
\]

when \( n \) is even \( (n = 2m) \), where \( d(\theta, \theta') \) is the spherical distance between \( \theta \) and \( \theta' \).

**Corollary 2.** (The radial wave equation in the spherical space one-dimensional) We let \( \mu \to \frac{1}{2} \) in Theorem 3. We obtain the solution of the Cauchy problem for the radial wave equation (see Theorem 2).

### 8. Numerical Trials

**Example.** When \( \nu = -\frac{1}{2} \) the radial wave problem

\[
(P) \begin{cases}
\left( \frac{\partial^2}{\partial x^2} + 2 \cot x \frac{\partial}{\partial x} - 1 \right) U(t, x) = \frac{\partial^2}{\partial t^2} U(t, x), \\
U(0, x) = 0, \quad U_t(0, x) = \sin x
\end{cases}
\]

has a unique solution given by

\[
U(t, x) = \frac{2t - \sin(2t) \cos(2x)}{4 \sin x}.
\]

We compare the exact solution with the approximate solution obtained by discretization of an interval \([A, B]\), \( A > 0 \) with a step \( \Delta x \) and a discretization of time with a step \( \Delta t \) (see Figure 1).

Let \( x_j = A + j\Delta x \), \( 1 \leq j \leq n_x \), \( L = B - A \), \( \Delta x = L/(n_x + 1) \) and \( t_n = n_t\Delta t \).

Numerically solving the problem \((P)\) means finding a discrete function \( U \) defined in points \((x_j, t_n)\), we note \( U^n_j \) the values of \( U \) at these points. The function \( U \) is obtained as the solution of a discrete problem

\[
\left[ 1 - \theta - (1 - \eta)R_j \right] U^n_{j-1} + [2(1 - \theta) + r_1] U^n_j + [1 - \theta + (1 - \eta)R_j] U^{n+1}_{j+1}
\]

\[
= (\eta R_j - \theta) U^n_{j-1} + (2\theta - 2r_1 - r_2) U^n_j - (\theta + \eta R_j) U^{n+1}_{j+1} + \eta U^{n-1}_j
\]

\[
U^n_0 = 0, \quad U^n_{-1} = -(\Delta t) g(x_j),
\]

where \( r_1 = \frac{(\Delta x)^2}{(\Delta t)^2} \), \( r_2 = -(\Delta x)^2 \) and \( R_j = \frac{\Delta x}{\tan(x_j)} \). (We take \( A = 0.5 \), \( B = 3 \), \( n_x = 10 \), \( n_t = 30 \), \( \Delta t = 0.01 \) and \( \theta = \eta = 0.5 \).)
Figure 1: Representation of the two solutions to the radial wave problem

References


