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FATOU PROPERTY OF PREDUAL MORREY SPACES WITH NON-DOUBLING MEASURES

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Abstract: The aim of this paper is to establish that the predual Morrey spaces are closed under taking increasing limit. As an application, the boundedness property of the fractional integral operators on predual Morrey spaces is obtained. This result answers the (open) question from our earlier paper "Predual spaces of Morrey spaces with non-doubling measures". This property will be used to supplement that paper, where the boundedness property of the fractional integral operators is missing.

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1. Introduction

This paper concerns a property of function spaces related to the Morrey spaces. The observation made by C. Morrey in 1938 has become a useful tool for partial differential equations, see [6]. Nowadays, his technique turned out to be a wide

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theory of function spaces called Morrey spaces. The Morrey spaces are now used in several branches of mathematics such as PDE and potential theory. Later, apart from the connection with PDE, many researchers considered the Morrey spaces from the viewpoint of geometry. In [7], the authors defined and investigated Morrey spaces when we are given a Radon measure μ.

First, let us recall some notations and definitions to define the Morrey spaces on $\mathbb{R}^d$ equipped with the underlying Radon measure μ. We do not assume the "so-called" doubling condition on μ; we do not necessarily suppose that there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \mathbb{R}^d$ and $r > 0$. By a "cube" we mean a closed cube whose edges are parallel to the coordinate axes. Its side length is denoted by $\ell(Q)$ and its center by $z(Q)$. For $c > 0$, cQ denotes a cube concentric to Q with sidelength $c\,\ell(Q)$. The set of all cubes $Q \subset \mathbb{R}^d$ satisfying $0 < \mu(Q) < \infty$ is denoted by $\mathcal{Q}(\mu)$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent number of p.

Recall that, for $1 \le q \le p < \infty$, the Morrey space $\mathcal{M}^p_q(\mu)$ is the set of all μ-measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^p_q(\mu)} = \|f\|_{\mathcal{M}^p_q(2,\mu)} = \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}$$

is finite, see [7, p. 1535].

We denote by $\mathcal{D}$ the family of all dyadic cubes of the form $Q = 2^{-k}(m + [0,1)^d)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^d$. We can show that, for any $k > 1$, the norms

$$\|f\|_{\mathcal{M}^p_q(k,\mu)} = \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}$$

and

$$\|f\|_{\mathcal{M}^p_q(k,\mu,\mathcal{D})} = \sup_{Q \in \mathcal{D}} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}$$

are equivalent to the norm $\|f\|_{\mathcal{M}^p_q(\mu)}$.

Based on this observation, in [8, Definition 3.2] we defined the predual of $\mathcal{M}^p_q(\mu)$.

Definition 1.1. Let $1 \le p \le q < \infty$. An $\mathcal{H}^p_q(\mu)$-block is a μ-measurable function b supported on a dyadic cube Q such that

$$\|b\|_{L^q(\mu)} \le \mu(2Q)^{\frac{1}{q}-\frac{1}{p}}.$$

The space $\mathcal{H}^p_q(\mu)$ is defined by the set of all functions $f \in L^p(\mu)$ such that there exist a complex sequence $\{\lambda_Q\}_{Q\in\mathcal{D}}$ and a collection of $\mathcal{H}^p_q(\mu)$-blocks $\{b_Q\}_{Q\in\mathcal{D}}$ such that

$$f = \sum_{Q\in\mathcal{D}} \lambda_Q f_Q \tag{1}$$

in the topology of $L^p(\mu)$.

For such f define the norm $\|f\|_{\mathcal{H}^p_q(\mu)}$ by

$$\|f\|_{\mathcal{H}^p_q(\mu)} \equiv \inf \left\{ \|\{\lambda_Q\}\|_{\ell^1(\mathcal{D})} : f = \sum_{Q\in\mathcal{D}} \lambda_Q b_Q \right\} < \infty,$$

where $\|\{\lambda_Q\}\|_{\ell^1(\mathcal{D})} = \sum_{Q\in\mathcal{D}} |\lambda_Q|$ and each b_Q is an $\mathcal{H}^p_q(\mu)$-block supported on the dyadic cube Q and the infimum is taken over all possible decompositions of f.

An elementary observation made in [8] is the following relation:

Proposition 1.2. *Let $1 < p \le q < \infty$. Then the dual of $\mathcal{H}^p_q(\mu)$ is $\mathcal{M}^{p'}_{q'}(\mu)$ in the following sense:*

1. *Let $f \in \mathcal{M}^{p'}_{q'}(\mu)$. Then $g \cdot f \in L^1(\mu)$ for any $g \in \mathcal{H}^p_q(\mu)$ and the mapping*

$$L_f : g \in \mathcal{H}^p_q(\mu) \mapsto \int_{\mathbb{R}^d} f(x)g(x)\, d\mu(x)$$

is a bounded linear functional on $\mathcal{H}^p_q(\mu)$. The operator norm of L_f is given by;

$$\|L_f\|_{\mathcal{H}^p_q(\mu)^*} = \|f\|_{\mathcal{M}^{p'}_{q'}(\mu)}.$$

2. *Conversely, any bounded linear functional L on $\mathcal{H}^p_q(\mu)$ is realized as $L = L_f$ by a certain $f \in \mathcal{M}^{p'}_{q'}(\mu)$.*

When $\mu = dx$, the space $\mathcal{H}^p_q(\mu)$ was investigated by Zorko [10]; see [5] as well. We refer to [1] and [2] for more recent characterizations.

The main theorems in this paper are the following results, which answers [3, Problem 11.4] affirmatively.

Theorem 1.3. *Let $1 < p \le q < \infty$. If $\{f_k\}_{k=1}^{\infty}$ is a norm-bounded sequence of $\mathcal{H}_q^p(\mu)$ satisfying*

$$0 \le f_1 \le f_2 \le \cdots,$$

then $f \equiv \lim_{j\to\infty} f_j \in \mathcal{H}_q^p(\mu)$ and $\|f\|_{\mathcal{H}_q^p(\mu)} \le \sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{H}_q^p(\mu)}$.

An equivalent form of Theorem 1.3 is the following result, which is recorded as [3, Problem 11.4].

Theorem 1.4. *Let $1 < p \le q < \infty$. If g is a μ-measurable function such that*

$$M = \sup\left\{\int_{\mathbb{R}^d} |f(x)g(x)|\,d\mu(x) \,:\, f \in \mathcal{M}_{q'}^{p'}(\mu),\, \|f\|_{\mathcal{M}_{q'}^{p'}(\mu)} = 1\right\} < \infty, \tag{2}$$

then $g \in \mathcal{H}_q^p(\mu)$ and

$$M = \|g\|_{\mathcal{H}_q^p(\mu)}. \tag{3}$$

What had been difficult is the fact that (2) by no means guarantees that $g \in \mathcal{H}_q^p(\mu)$. More precisely, we could not say anything about the decomposition (1) from (3). To overcome this difficulty, we propose an equivalent expression (4) in Section 2. Due to the generalized setting, we find that the norm

$$\|f\|_{\mathcal{M}_q^p(\mu,\mathcal{D})} = \|f\|_{\mathcal{M}_q^p(2,\mu,\mathcal{D})} = \sup\left\{\mu(2Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f(y)|^q\,d\mu(y)\right)^{\frac{1}{q}} \,:\, Q \in \mathcal{D}\right\}$$

is not sufficient. This surely works out for the case when $\mu = dx$ [9]. Recall that $\mathcal{D}$ is countable. As the proof shows that this fact and the reflexibity of $L^p(\mu)$ with $p \in (1,\infty)$ pave the way of construction of the decomposition (1). This technique originally dates back to [4].

Here we describe the organization of the present paper. We prove Theorems 1.3 and 1.4 in Sections 2 and 3, respectively. In Section 4, we give an application of Theorem 1.4.

2. Proof of Theorem 1.3

In what follows the letter C will be used to denote constants that may change from one occurrence to another.

We know that the number "2" appearing in the above definition can be replaced with any number in $(1,\infty)$: this will yield an equivalent norm.

Definition 2.1. Define $\mathcal{D}(\mu)$ as follows:

1. Set $M(\mu) \equiv \{x \in \mathbb{R}^d : \mu(\{x\}) > 0\}$. For $x \in M(\mu)$, we choose a dyadic cube $Q_x \in \mathcal{D}$ so that $\mu(2Q_x) \le 2\mu(\{x\})$ and that $x \in Q_x$. Define

$$\mathcal{D}_0(\mu) \equiv \{Q_x : x \in M(\mu)\}$$

and define

$$\mathcal{D}_1(\mu) \equiv \bigcup_{x \in M(\mu)} \{Q' \in \mathcal{D} : x \in Q' \subsetneq Q_x\}.$$

2. (a) Let $\mu(\mathbb{R}^d) = \infty$. Define $\mathcal{D}(\mu)$ by

$$\mathcal{D}(\mu) \equiv \mathcal{D} \setminus \mathcal{D}_1(\mu).$$

   (b) Let $\mu(\mathbb{R}^d) < \infty$. Define

$$\mathcal{D}_2(\mu) \equiv \{Q \in \mathcal{D} : 2\mu(2Q) \ge \mu(\mathbb{R}^d)\}$$

   and define $\mathcal{D}(\mu)$ by

$$\mathcal{D}(\mu) \equiv (\mathcal{D} \cup \{\mathbb{R}^d\}) \setminus \mathcal{D}_1(\mu) \setminus \mathcal{D}_2(\mu).$$

   In this situation, we consider any $L^q(\mu)$-function b with norm less than $\mu(\mathbb{R}^d)^{\frac{1}{q}-\frac{1}{p}}$ as a (special) $\mathcal{H}^p_q(\mu)$-block.

The following observation is a key to our proof.

Lemma 2.2. *The following norm is equivalent to the original norm of $\mathcal{H}^p_q(\mu)$. For $f \in L^p(\mu)$, we let*

$$\|f\|_{\mathcal{H}^p_q(\mathcal{D}(\mu))} \equiv \inf \sum_{Q \in \mathcal{D}(\mu)} |\lambda_Q|, \tag{4}$$

where the infimum is taken over all expressions

$$f = \sum_{Q \in \mathcal{D}(\mu)} \lambda_Q b_Q \tag{5}$$

satisfying

$$\{\lambda_Q\} \in \ell^1(\mathcal{D}(\mu)), \quad \operatorname{supp} b_Q \subset Q, \quad \|b_Q\|_{L^q(\mu)} \le \mu(2Q)^{\frac{1}{q}-\frac{1}{p}}. \tag{6}$$

Proof. We suppose $\mu(\mathbb{R}^d) < \infty$. Otherwise, the proof will be somewhat simpler.

Let us suppose that $f \in \mathcal{H}^p_q(\mu)$. Then we have an expression;

$$f = \sum_{Q \in \mathcal{D}} \lambda_Q b_Q,$$

where each b_Q is an $\mathcal{H}^p_q(\mu)$-block supported on Q and

$$\sum_{Q \in \mathcal{D}} |\lambda_Q| \le 2\|f\|_{\mathcal{H}^p_q(\mu)}.$$

First, decompose f as

$$\begin{cases} f = f_1 + f_2 + f_3, \\ f_1 \equiv \sum_{Q \in \mathcal{D}_1(\mu)} \lambda_Q b_Q, \\ f_2 \equiv \sum_{Q \in \mathcal{D}(\mu)} \lambda_Q b_Q, \\ f_3 \equiv \sum_{Q \in \mathcal{D}_2(\mu)} \lambda_Q b_Q. \end{cases}$$

We suppose

$$\sum_{Q \in \mathcal{D}_1(\mu)} |\lambda_Q|, \sum_{Q \in \mathcal{D}(\mu)} |\lambda_Q|, \sum_{Q \in \mathcal{D}_2(\mu)} |\lambda_Q| > 0$$

for the sake of simplicity. Otherwise the proof becomes a little simpler.

We divide $\mathcal{D}_1(\mu)$ into the disjoint sets $\mathcal{D}_Q(\mu)$, $Q \in \mathcal{D}_0(\mu)$, as

$$\mathcal{D}_1(\mu) = \bigcup_{Q \in \mathcal{D}_0(\mu)} \mathcal{D}_Q(\mu)$$

and $\mathcal{D}_Q(\mu)$ fulfills

$$\operatorname{supp} b_{Q'} \subset Q \text{ and } \mu(2Q) \le 2\mu(2Q') \text{ when } Q' \in \mathcal{D}_Q(\mu),$$

where we have used the fact that, if $x \in Q' \subset Q_x$, $x \in M(\mu)$, one has

$$0 < \mu(\{x\}) \le \mu(2Q') \le \mu(2Q_x) \le 2\mu(\{x\}) \le 2\mu(2Q').$$

We now rewrite f_1 as

$$\begin{aligned} f_1 &= \sum_{Q \in \mathcal{D}_0(\mu)} \sum_{Q' \in \mathcal{D}_Q(\mu)} \lambda_{Q'} b_{Q'} \\ &= \sum_{Q \in \mathcal{D}_0(\mu)} \left\{ \sum_{Q' \in \mathcal{D}_Q(\mu)} |\lambda_{Q'}| \right\} \left\{ \sum_{Q' \in \mathcal{D}_Q(\mu)} |\lambda_{Q'}| \right\}^{-1} \sum_{Q' \in \mathcal{D}_Q(\mu)} \lambda_{Q'} b_{Q'}. \end{aligned}$$

It follows that

$$\sum_{Q\in\mathcal{D}_0(\mu)}\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\le\sum_{Q\in\mathcal{D}}|\lambda_Q|\le 2\|f\|_{\mathcal{H}^p_q(\mu)}$$

and that, for $Q\in\mathcal{D}_0$,

$$\begin{aligned}&\left\{\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\right\}^{-1}\left\|\sum_{Q'\in\mathcal{D}_Q(\mu)}\lambda_{Q'}b_{Q'}\right\|_{L^q(\mu)}\\&\le\left\{\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\right\}^{-1}\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\|b_{Q'}\|_{L^q(\mu)}\\&\le\left\{\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\right\}^{-1}\sum_{Q'\in\mathcal{D}_Q(\mu)}|\lambda_{Q'}|\mu(2Q')^{\frac{1}{q}-\frac{1}{p}}\\&\le C\mu(2Q)^{\frac{1}{q}-\frac{1}{p}},\end{aligned}$$

which implies that C^{-1} times the left-hand side of the inequality is an $\mathcal{H}^p_q(\mu)$-block supported on Q. Since $\mathcal{D}_0(\mu)\subset\mathcal{D}(\mu)$, we have

$$\|f_1\|_{\mathcal{H}^p_q(\mathcal{D}(\mu))}\le C\|f\|_{\mathcal{H}^p_q(\mu)}.$$

Similarly,

$$f_3=\left\{\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\right\}\left\{\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\right\}^{-1}\sum_{Q\in\mathcal{D}_2(\mu)}\lambda_Q b_Q.$$

It follows that

$$\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\le 2\|f\|_{\mathcal{H}^p_q(\mu)}$$

and that

$$\begin{aligned}&\left\{\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\right\}^{-1}\left\|\sum_{Q\in\mathcal{D}_2(\mu)}\lambda_Q b_Q\right\|_{L^q(\mu)}\\&\le\left\{\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\right\}^{-1}\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\|b_Q\|_{L^q(\mu)}\\&\le\left\{\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\right\}^{-1}\sum_{Q\in\mathcal{D}_2(\mu)}|\lambda_Q|\mu(2Q)^{\frac{1}{q}-\frac{1}{p}}\le C\mu(\mathbb{R}^d)^{\frac{1}{q}-\frac{1}{p}},\end{aligned}$$

which implies that C^{-1} times the left-hand side of the inequality is a special $\mathcal{H}^p_q(\mu)$-block. Thus,

$$\|f_3\|_{\mathcal{H}^p_q(\mathcal{D}(\mu))}\le C\|f\|_{\mathcal{H}^p_q(\mu)}.$$

These complete the proof. □

With this observation in mind, we prove Theorem 1.3. Since the result follows readily from the monotone convergence theorem, we can suppose $p\ne q$. Also, we can suppose $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{H}^p_q(\mu)}<\infty$.

We write

$$f_k=\sum_{Q\in\mathcal{D}(\mu)}\lambda_{Q,k}b_{Q,k},$$

where each $b_{Q,k}$ is an $\mathcal{H}^p_q(\mu)$-block and

$$\sup_{k\in\mathbb{N}}\sum_{Q\in\mathcal{D}(\mu)}|\lambda_{Q,k}|\le 2\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{H}^p_q(\mu)}.$$

If we pass to a subsequence, we may assume that

$$\lambda_Q=\lim_{k\to\infty}\lambda_{Q,k},\quad b_Q=\lim_{k\to\infty}b_{Q,k}$$

exists for all $Q\in\mathcal{D}(\mu)$ in the topology of $\mathbb{C}$ and in the weak topology of $L^q(\mu)$, respectively. Since

$$\sum_{Q\in\mathcal{D}(\mu)}|\lambda_Q|\le 2\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{H}^p_q(\mu)},$$

$g\equiv\sum_{Q\in\mathcal{D}(\mu)}\lambda_Q b_Q\in\mathcal{H}^p_q(\mu)$. Let us check that $g=f$. To this end, we take a dyadic cube Q_0 and we prove

$$\int_{Q_0}f(x)\,d\mu(x)=\int_{Q_0}g(x)\,d\mu(x).$$

Once this is proved, by the Lebesgue differentiation theorem, we obtain $f(x) = g(x)$ for μ-a.e..

By the monotone convergence theorem, the matters are reduced to proving:

$$\lim_{k\to\infty}\int_{Q_0} f_k(x)\,d\mu(x) = \int_{Q_0} g(x)\,d\mu(x).$$

This follows once we prove

$$\lim_{k\to\infty}\int_{Q_0}\sum_{Q\supsetneq Q_0, Q\in\mathcal{D}(\mu)}\lambda_{Q,k}b_{Q,k}(x)\,d\mu(x) = \int_{Q_0}\sum_{Q\supsetneq Q_0, Q\in\mathcal{D}(\mu)}\lambda_Q b_Q(x)\,d\mu(x) \quad (7)$$

$$\lim_{k\to\infty}\int_{Q_0}\sum_{Q\subset Q_0, Q\in\mathcal{D}(\mu)}\lambda_{Q,k}b_{Q,k}(x)\,d\mu(x) = \int_{Q_0}\sum_{Q\subset Q_0, Q\in\mathcal{D}(\mu)}\lambda_Q b_Q(x)\,d\mu(x). \quad (8)$$

As for (7), when $\mu(\mathbb{R}^d) < \infty$, the sum $\displaystyle\sum_{Q\supsetneq Q_0, Q\in\mathcal{D}(\mu)}\lambda_{Q,k}b_{Q,k}$ is a finite sum so that (7) is easy.

When $\mu(\mathbb{R}^d) = \infty$, we fix $\varepsilon > 0$. Then

$$\begin{aligned}
&\left|\int_{Q_0}\sum_{\substack{Q\in\mathcal{D}(\mu),\\ Q\supsetneq Q_0, \mu(2Q)>\varepsilon^{-1}}}\lambda_{Q,k}b_{Q,k}(x)\,d\mu(x)\right|\\
&\le \sum_{\substack{Q\in\mathcal{D}(\mu),\\ Q\supsetneq Q_0, \mu(2Q)>\varepsilon^{-1}}}|\lambda_{Q,k}|\mu(Q_0)^{\frac{1}{q'}}\|b_{Q,k}\|_{L^q(\mu)}\\
&\le \sum_{\substack{Q\in\mathcal{D}(\mu),\\ Q\supsetneq Q_0, \mu(2Q)>\varepsilon^{-1}}}|\lambda_{Q,k}|\mu(Q_0)^{\frac{1}{q'}}\mu(2Q)^{\frac{1}{q}-\frac{1}{p}}\\
&\le 2\varepsilon^{\frac{1}{p}-\frac{1}{q}}\mu(Q_0)^{\frac{1}{q'}}\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{H}^p_q(\mu)}.
\end{aligned}$$

The same estimate holds if we replace $b_{Q,k}$ with b_k. Thus,

$$\int_{Q_0}\sum_{\substack{Q\in\mathcal{D}(\mu),\\ Q\supset Q_0, \mu(Q)>\varepsilon^{-1}}}\lambda_{Q,k}b_{Q,k}(x)\,d\mu(x) = o(1)$$

$$\int_{Q_0}\sum_{\substack{Q\in\mathcal{D}(\mu),\\ Q\supset Q_0, \mu(Q)>\varepsilon^{-1}}}\lambda_Q b_Q(x)\,d\mu(x) = o(1)$$

as $\varepsilon \to 0$, where the implicit constant in the Landau symbol is uniform over k. Meanwhile, since

$$\int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \supset Q_0, \mu(Q) \le \varepsilon^{-1}}} \lambda_{Q,k} b_{Q,k}$$

is a finite sum, it follows that

$$\lim_{k \to \infty} \int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \supset Q_0, \mu(Q) \le \varepsilon^{-1}}} \lambda_{Q,k} b_{Q,k}(x) \, d\mu(x) = \int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \supset Q_0, \mu(Q) \le \varepsilon^{-1}}} \lambda_Q b_Q(x) \, d\mu(x).$$

As for (8), we fix $\varepsilon > 0$ arbitrarily.

$$\begin{aligned}
\left| \int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} \lambda_{Q,k} b_{Q,k}(x) \, d\mu(x) \right| &= \left| \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} \int_{Q_0} \lambda_{Q,k} b_{Q,k}(x) \, d\mu(x) \right| \\
&= \left| \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} \int_{Q} \lambda_{Q,k} b_{Q,k}(x) \, d\mu(x) \right| \\
&\le \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} |\lambda_{Q,k}| \mu(Q)^{\frac{1}{q'}} \|b_{Q,k}\|_{L^q} \\
&\le \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} |\lambda_{Q,k}| \mu(Q)^{\frac{1}{q'}} \mu(2Q)^{\frac{1}{q} - \frac{1}{p}} \\
&\le 2\varepsilon^{\frac{1}{p'}} \sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{H}^p_q(\mu)}.
\end{aligned}$$

Thus,

$$\int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} \lambda_{Q,k} b_{Q,k}(x) \, d\mu(x) = o(1)$$

$$\int_{Q_0} \sum_{\substack{Q \in \mathcal{D}(\mu), \\ Q \subset Q_0, \mu(Q) < \varepsilon}} \lambda_Q b_Q(x) \, d\mu(x) = o(1)$$

as $\varepsilon \to 0$, where the implicit constant in the Landau symbol is uniform over k. Since

$$\int_{Q_0} \sum_{Q \subset Q_0, Q \in \mathcal{D}(\mu), \mu(Q) \geq \varepsilon} \lambda_{Q,k} b_{Q,k}$$

is a finite sum, it follows that

$$\lim_{k \to \infty} \int_{Q_0} \sum_{Q \subset Q_0, Q \in \mathcal{D}(\mu), \mu(Q) \geq \varepsilon} \lambda_{Q,k} b_{Q,k}(x)\, d\mu(x) = \int_{Q_0} \sum_{Q \subset Q_0, Q \in \mathcal{D}(\mu), \mu(Q) \geq \varepsilon} \lambda_Q b_Q(x)\, d\mu(x).$$

Thus, (8) is proved.

3. Proof of Theorem 1.4

As we have mentioned in Section 1, the heat of the matter is to prove $g \in \mathcal{H}^p_q(\mu)$; once this is proved, then we can invoke the Hahn-Banach theorem. To this end, by the decomposition, $g = (\mathrm{Re} g)_+ - (\mathrm{Im} g)_- + i(\mathrm{Im} g)_+ - i(\mathrm{Im} g)_-$ we can suppose that g is non-negative. Let us set $g_j = g\chi_{\{|g| \leq j\}}\chi_{\{|x| \leq j\}}$ for $j = 1, 2, \ldots$. Since g_j is a bounded function with compact support, $g_j \in \mathcal{H}^p_q(\mu)$. By (2) and the Hahn-Banach theorem shows that $\|g_j\|_{\mathcal{H}^p_q(\mu)} \leq M$. Since g_j is increasing, we are in the position of using Theorem 1.3. Thus, $g \in \mathcal{H}^p_q(\mu)$ and the proof is complete.

4. An Application–Boundedness of Fractional Integral Operators

As an application, we shall investigate the property of the fractional integral operators. Here, for the sake of simplicity, we suppose that there exists a constant $C > 0$ such that $\mu(B(x, r)) \leq Cr^n$ for some $0 < n \leq d$. Let $0 < \alpha < n$. Then define the fractional integral operator of order α by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{n-\alpha}}\, d\mu(y).$$

In [7, Theorem 3.3], we established the following

Proposition 4.1. *Let $1 < q \le p < \infty$, $1 < t \le s < \infty$. Assume*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then I_α is bounded from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$.

As an application of Theorem 1.4 and Proposition 4.1, we can prove the following result:

Theorem 4.2. *Let $1 < q \le p < \infty$, $1 < t \le s < \infty$. Assume*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then I_α is bounded from $\mathcal{H}^{s'}_{t'}(\mu)$ to $\mathcal{H}^{p'}_{q'}(\mu)$.

Proof. We need to prove that

$$\|I_\alpha f\|_{\mathcal{H}^{p'}_{q'}(\mu)} \le C\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)}$$

for all $f \in \mathcal{H}^{s'}_{t'}(\mu)$.

Notice that $f \in \mathcal{H}^{s'}_{t'}(\mu)$ implies $|f| \in \mathcal{H}^{s'}_{t'}(\mu)$ and that

$$\| \, |f| \, \|_{\mathcal{H}^{s'}_{t'}(\mu)} \le C\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)}.$$

Thus, we may suppose that f is non-negative. Also, a routine trunction technique allows us to assume that f is compactly supported.

By Theorem 1.4, we can reduce the matters to the estimate;

$$\|I_{\alpha,N} f\|_{\mathcal{H}^{p'}_{q'}(\mu)} \le C\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)},$$

where

$$I_{\alpha,N} f(x) = \int_{N^{-1}<|x-y|<N} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y)$$

and C is a constant independent of N. Since f is compactly supported, $I_{\alpha,N} f$ is compactly supported and μ-essentially bounded. Thus, at least, we have $I_{\alpha,N} f \in \mathcal{H}^{p'}_{q'}(\mu)$.

To obtain quantitative information, we argue by using the Hahn-Banach theroem as follows:

$$\begin{aligned}
\|I_{\alpha,N}f\|_{\mathcal{H}^{p'}_{q'}(\mu)} &= \sup\left\{\left|\int_{\mathbb{R}^d} g(x)I_{\alpha,N}f(x)\,d\mu(x)\right| \,:\, g\in\mathcal{M}^p_q(\mu), \|g\|_{\mathcal{M}^p_q(\mu)}=1\right\}\\
&= \sup\left\{\left|\int_{\mathbb{R}^d} f(x)I_{\alpha,N}g(x)\,d\mu(x)\right| \,:\, g\in\mathcal{M}^p_q(\mu), \|g\|_{\mathcal{M}^p_q(\mu)}=1\right\}\\
&\le \sup\left\{\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)}\|I_{\alpha,N}g\|_{\mathcal{M}^s_t(\mu)} \,:\, g\in\mathcal{M}^p_q(\mu), \|g\|_{\mathcal{M}^p_q(\mu)}=1\right\}\\
&\le C\sup\left\{\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)}\|g\|_{\mathcal{M}^p_q(\mu)} \,:\, g\in\mathcal{M}^p_q(\mu), \|g\|_{\mathcal{M}^p_q(\mu)}=1\right\}\\
&= C\|f\|_{\mathcal{H}^{s'}_{t'}(\mu)}.
\end{aligned}$$

Here we used Propositions 1.2 and 4.1 for the first and second inequalities, respectively. □

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