POSITIVE SOLUTIONS FOR SINGULAR SYSTEMS OF THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

Yatao Du¹ §, Hanying Feng², Xingfang Feng³
¹,²,³Department of Mathematics
Shijiazhuang Mechanical Engineering College
P.R. CHINA

Abstract: In this paper, we study the existence of positive solutions of a three-point boundary value problem for a system of third-order nonlinear singular differential equations by the Krasnoselskii fixed point theorem.

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1. Introduction

Singular boundary value problems arise from many fields in physics, biology and chemistry, and play a very important role in both theoretical development and application. Many attempts have been made to study a variety of nonlinear singular differential equations or systems. However, most works only deal with boundary value problems for second-order or fourth-order nonlinear singular systems (see [1],[3],[4],[6]-[9]). Very few papers studied third-order nonlinear singular systems.

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§Correspondence author
In this paper, we study the following singular systems of nonlinear third-order three-point boundary value problems (BVP)

\[
\begin{cases}
-u'' = f(t, v), & t \in (0, 1), \\
-v'' = g(t, u), & t \in (0, 1), \\
u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \\
v(0) = v'(0) = 0, & v'(1) = \alpha v'(\eta),
\end{cases}
\]

(1.1)

where \(0 < \eta < 1\) and \(1 < \alpha < \frac{1}{\eta}\), \(f, g \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)\), \(f(t, 0) \equiv 0, g(t, 0) \equiv 0, t \in (0, 1)\), and \(f, g\) may be singular at \(t = 0\) and/or \(t = 1\), in which \(\mathbb{R}^+ = [0, +\infty)\). The existence of positive solutions to BVP (1.1) is obtained under some weaker assumptions that \(f\) or \(g\) does not possess any growth, sublinear or superlinear conditions.

The vector \((u, v) \in C^3(0, 1) \times C^3(0, 1)\) is said to be a positive solution of BVP (1.1) if \((u, v)\) satisfies (1.1) and \(u(t) > 0, v(t) > 0\) for \(t \in (0, 1)\).

For convenience, we make the following assumptions:

(H1) \(f \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+), g \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)\) and there exist \(p_i \in C((0, 1), \mathbb{R}^+), q_i \in C((0, 1), \mathbb{R}^+), i = 1, 2\), \(q_2(0) = 0\) such that

\[f(t, x) \leq p_1(t)q_1(x), \quad g(t, y) \leq p_2(t)q_2(y), \quad t \in (0, 1), x, y \in \mathbb{R}^+\]

and

\[a = \int_0^1 s(1-s)p_1(s)ds < +\infty, \quad b = \int_0^1 s(1-s)p_2(s)ds < +\infty.\]

(H2) There exist \(r_1, r_2 \in (0, +\infty)\) with \(r_1 r_2 \geq 1\) such that

\[\limsup_{x \to 0^+} \frac{q_1(x)}{x^{r_1}} < +\infty, \quad \limsup_{y \to 0^+} \frac{q_2(y)}{y^{r_2}} = 0.\]

(H3) There exist \(l_1, l_2 \in (0, +\infty)\) with \(l_1 l_2 \geq 1\) such that

\[\liminf_{x \to +\infty} \min_{t \in [\frac{1}{\alpha}, \eta]} \frac{f(t, x)}{x^{l_1}} > 0, \quad \liminf_{y \to +\infty} \min_{t \in [\frac{1}{\alpha}, \eta]} \frac{g(t, y)}{y^{l_2}} = +\infty.\]

(H4) There exist \(\alpha_1, \alpha_2 \in (0, +\infty)\) with \(\alpha_1 \alpha_2 \leq 1\) such that

\[\limsup_{x \to +\infty} \frac{q_1(x)}{x^{\alpha_1}} < +\infty, \quad \limsup_{y \to +\infty} \frac{q_2(y)}{y^{\alpha_2}} = 0.\]

(H5) There exist \(\beta_1, \beta_2 \in (0, +\infty)\) with \(\beta_1 \beta_2 \leq 1\) such that

\[\liminf_{x \to 0^+} \min_{t \in [\frac{1}{\alpha}, \eta]} \frac{f(t, x)}{x^{\beta_1}} > 0, \quad \liminf_{y \to 0^+} \min_{t \in [\frac{1}{\alpha}, \eta]} \frac{g(t, y)}{y^{\beta_2}} = +\infty.\]
(H₆) There exists $N > 0$ such that

$$
\sup_{u \in [0, Ma_1 b]} q_1(u) \leq \frac{N}{aa_1},
$$

where $M = \sup_{u \in [0, N]} q_2(u)$, $a_1 = \frac{1+\alpha}{1-\alpha \eta}$, and $a, b$ are the same as in (H₁).

The main results obtained are as follows:

**Theorem 1.1.** Assume that (H₁) – (H₃) hold. Then BVP (1.1) has at least one positive solution.

**Theorem 1.2.** Assume that (H₁), (H₄) and (H₅) hold. Then BVP (1.1) has at least one positive solution.

**Theorem 1.3.** Assume that (H₁), (H₃), (H₅) and (H₆) hold. Then BVP (1.1) has at least two positive solutions.

2. Preliminaries and lemmas

**Lemma 2.1.** (see [2]) Let $E$ be a Banach space and $K \subset E$ be a cone. Assume $\Omega_1$ and $\Omega_2$ are open subsets of $E$ with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that one of the following conditions is satisfied:

(i) $\|Au\| \leq \|u\|$, for $u \in K \cap \partial \Omega_1$; $\|Au\| \geq \|u\|$, for $u \in K \cap \partial \Omega_2$;

(ii) $\|Au\| \leq \|u\|$, for $u \in K \cap \partial \Omega_2$; $\|Au\| \geq \|u\|$, for $u \in K \cap \partial \Omega_1$.

Then $A$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

**Lemma 2.2.** (see [5]) Let $\alpha \eta \neq 1$. Then for $y \in C[0,1]$, the BVP

$$
\begin{cases}
  u'''(t) + y(t) = 0, & t \in (0,1), \\
  u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta)
\end{cases}
$$

is satisfied:
has a unique solution \( u(t) = \int_0^1 G(t, s)y(s)ds \), where

\[
G(t, s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} 
(2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha-1), & s \leq \min\{\eta, t\}, \\
t^2(1-\alpha\eta) + t^2 s(\alpha-1), & t \leq s \leq \eta, \\
(2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\
t^2(1-s), & \max\{\eta, t\} \leq s.
\end{cases}
\]

Lemma 2.3. (see [5]) Let \( 0 < \eta < 1 \) and \( 1 < \alpha < \frac{1}{\eta} \). Then for any \((t, s) \in [0, 1] \times [0, 1]\),

\[0 \leq G(t, s) \leq a_1 s(1-s),\]

where \( a_1 = \frac{1+\alpha}{1-\alpha\eta} \).

Lemma 2.4. (see [5]) Let \( 0 < \eta < 1 \) and \( 1 < \alpha < \frac{1}{\eta} \). Then for any \((t, s) \in [\frac{\eta}{\alpha}, \eta] \times [0, 1]\),

\[G(t, s) \geq a_2 s(1-s),\]

where

\[0 < a_2 = \frac{\eta^2}{2\alpha^2(1-\alpha\eta)} \min\{\alpha-1, 1\} < 1.\]

Let \( E = C[0, 1] \) be a Banach space endowed with norm \( \|u\| = \max_{t \in [0, 1]} |u(t)| \). Let \( P = \{u \in E \mid u(t) \geq 0, t \in [0, 1]\} \). Then \( P \) is a cone of \( E \). For any \( r > 0 \), let \( B_r = \{u \in C[0, 1] \mid \|u\| < r\} \), \( \partial B_r = \{u \in C[0, 1] \mid \|u\| = r\} \).

It is easy to see that \((u, v) \in C^3(0, 1) \times C^3(0, 1)\) is a solution of BVP(1.1) if and only if \((u, v) \in C^2(0, 1) \times C^2(0, 1)\) is a solution of the following system of nonlinear integral equations:

\[
\begin{cases}
    u(t) = \int_0^1 G(t, s)f(s, v(s))ds, \\
v(t) = \int_0^1 G(t, s)g(s, u(s))ds, \quad t \in [0, 1].
\end{cases}
\]

Obviously, the above system of nonlinear integral equations can be regarded as the following nonlinear integral equation:

\[
u(t) = \int_0^1 G(t, s)f \left(s, \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau\right)ds, \quad t \in [0, 1]. \tag{2.1}\]
Define an operator $A : P \to P$ by

$$
(Au)(t) = \int_0^1 G(t, s) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds, \ t \in [0, 1].
$$

(2.2)

It is easy to verify that if $x(t)$ is a fixed point of $A$ in $C^2(0, 1)$, then BVP (1.1) has one solution $(u, v)$,

$$
\begin{align*}
\left\{ \begin{array}{l}
  u(t) = x(t), \\
  v(t) = \int_0^1 G(t, s) g(s, x(s)) ds,
\end{array} \right. \quad t \in [0, 1].
\end{align*}
$$

**Lemma 2.5.** Assume that $(H_1)$ holds. Then $A : P \to P$ is completely continuous.

**Proof.** First it is easy to see that $A$ maps $P$ into $P$. Then we prove that $A$ maps bounded sets into bounded sets.

Suppose $D \subset P$ is an arbitrary bounded set. Then there exists $M_1 > 0$ such that $\|u\| \leq M_1$, for all $u \in D$. By the continuity of $q_2$, there is $M_2 > 0$ such that $M_2 = \sup_{x \in [0, M_1]} q_2(x)$. So for any $u \in D$, $s \in [0, 1]$, by Lemma 2.3, we have

$$
\int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \leq \int_0^1 G(s, \tau) p_2(\tau) q_2(u(\tau)) d\tau
$$

$$
\leq M_2 a_1 \int_0^1 \tau(1 - \tau) p_2(\tau) d\tau = M_2 a_1 b. \quad (2.3)
$$

By the continuity of $q_1$, there is $M_3 > 0$ such that $M_3 = \sup_{x \in [0, M_2 a_1 b]} q_1(x)$. Then from (2.3), $(H_1)$ and Lemma 2.3 that

$$
(Au)(t) = \int_0^1 G(t, s) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
$$

$$
\leq \int_0^1 G(t, s) p_1(s) q_1 \left( \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
$$

$$
\leq M_3 a_1 \int_0^1 s(1 - s) p_1(s) ds < +\infty. \quad (2.4)
$$

Therefore, $A(D)$ is uniformly bounded. In the following we show that $A(D)$ is equicontinuous. According to Lemma 2.2, we have

$$
(Au)'(t) = \int_0^t s(1 - t) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
$$

$$
+ \int_t^1 t(1 - s) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
$$
\[
+ \frac{\alpha t}{1 - \alpha \eta} \int_0^1 G_1(\eta, s) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
\]

where

\[
G_1(t, s) = \begin{cases} 
  s(1 - t), & 0 \leq s \leq t \leq 1, \\
  t(1 - s), & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Hence, by (H1), we have

\[
|(Au)'(t)| \leq \int_0^t s(1 - t)p_1(s) \left( \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
\]
\[
+ \int_t^1 t(1 - s)p_1(s) \left( \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
\]
\[
+ \frac{\alpha t}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)p_1(s)q_1 \left( \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
\]
\[
\leq M_3 \left( \int_0^t s(1 - t)p_1(s)ds + \int_t^1 t(1 - s)p_1(s)ds \\
+ \int_0^1 \frac{\alpha}{1 - \alpha \eta}G_1(\eta, s)p_1(s)ds \right). \tag{2.5}
\]

Let

\[
h(t) = \int_0^t s(1 - t)p_1(s)ds + \int_t^1 t(1 - s)p_1(s)ds,
\]

then

\[
\int_0^1 h(t)dt = \int_0^1 \int_s^1 s(1 - t)p_1(s)dt ds + \int_0^1 \int_0^s t(1 - s)p_1(s)dt ds
\]
\[
= \frac{1}{2} \int_0^1 s(1 - s)p_1(s)ds < +\infty.
\]

Notice that \( \int_0^1 G_1(\eta, s)p_1(s)ds \leq \int_0^1 s(1 - s)p_1(s)ds \). Let

\[
\mu(t) = h(t) + \int_0^1 \frac{\alpha}{1 - \alpha \eta}G_1(\eta, s)p_1(s)ds,
\]

then,

\[
\int_0^1 \mu(s)ds = \int_0^1 h(s)ds + \frac{\alpha}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)p_1(s)ds,
\]
\[
\leq \left( \frac{1}{2} + \frac{\alpha}{1 - \alpha \eta} \right) \int_0^1 s(1 - s)p_1(s)ds < +\infty. \tag{2.6}
\]
We know from \((H_1)\) and \((2.6)\) that \(\mu(t) \in L^1[0,1]\). Thus for any given \(0 \leq t_1 \leq t_2 \leq 1\) and \(\forall u \in D\), by \((2.5)\), we obtain

\[
|(Au)(t_1) - (Au)(t_2)| = \left| \int_{t_1}^{t_2} (Au)'(t)dt \right| \leq M_3 \int_{t_1}^{t_2} \mu(t)dt \quad (2.7)
\]

By virtue of \((2.6), (2.7)\), and the absolute continuity of the integral function, it follows that \(A(D)\) is equicontinuous. This together with \((2.4)\) and the Ascoli-Arzelà theorem guarantee that \(A(D)\) is relatively compact.

Now, we prove that \(A\) is continuous. Suppose \(u_m, u \in D\) and \(\|u_m - u\| \to 0\) \((m \to 0)\). Then there exists \(M_4 > 0\) such that \(\|u_m\| \leq M_4\) and \(\|u\| \leq M_4\). From the above proof we know that \(\{Au_m\}\) is relatively compact. In the following we prove \(\|Au_m - Au\| \to 0\) \((m \to 0)\). In fact, if this is not true, then there exists \(\varepsilon_0 > 0\) and \(\{u_{m_k}\} \subset \{u_m\}\) such that \(\|Au_{m_k} - Au\| \geq \varepsilon_0\) \((k = 1, 2, \cdots)\). Since \(\{Au_{m_k}\}\) is relatively compact, there exists a subsequence of \(\{Au_{m_k}\}\) which converges in \(P\) to some \(y \in P\). Without loss of generality, we assume that \(\{Au_{m_k}\}\) itself converges to \(y\), that is

\[
\lim_{k \to +\infty} \|Au_{m_k} - y\| = 0.
\]

Obviously, \((Au_{m_k})(t) \to y(t)\) as \(k \to +\infty\), for \(t \in [0,1]\). By \((H_1)\) and Lemma 2.3, we obtain

\[
G(s, \tau)g(\tau, u_{m_k}(\tau)) \leq a_1\tau(1-\tau)p_2(\tau)q_2(u_{m_k}(\tau)) \leq M_5a_1\tau(1-\tau)p_2(\tau), \quad s \in [0,1],
\]

where \(M_5 = \sup_{x \in [0,M_4]}q_2(x) < +\infty\). Hence,

\[
G(t,s)f \left( s, \int_0^1 G(s, \tau)g(\tau, u_{m_k}(\tau))d\tau \right) \leq a_1s(1-s)p_1(s)q_1 \left( \int_0^1 G(s, \tau)g(\tau, u_{m_k}(\tau))d\tau \right)
\]

\[
\leq M_6a_1s(1-s)p_1(s)\quad (2.8)
\]

where \(M_6 = \sup_{x \in [0,M_5a_1b]}q_1(x)\). Then \((H_1), (2.8)\) and Lebesgue Control Theorem imply that

\[
y(t) = \lim_{k \to +\infty} (Au_{m_k})(t) = (Au)(t), \quad t \in [0,1],
\]

that is, \(y = Au\). This is a contradiction with \(\|Au_{m_k} - Au\| \geq \varepsilon_0\) \((k = 1, 2, \cdots)\). Consequently, \(A\) is continuous on \(P\). To sum up, Lemma 2.5 is proved. \(\square\)
Let
\[ K = \left\{ u \in P \mid \min_{t \in \left[ \frac{\alpha}{\eta}, \eta \right]} u(t) \geq \gamma \|u\| \right\}, \tag{2.9} \]
where \( \gamma = \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha - 1, 1\} \). It is obvious that \( K \) is a subcone of \( P \).

**Lemma 2.6.** \( AK \subset K \).

**Proof.** For any \( u \in K \), we prove \( Au \in K \). According to Lemma 2.3, we know
\[
\|Au\| = \max_{t \in [0,1]} (Au)(t) \leq a_1 \int_0^1 s(1-s)f\left(s, \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau\right) ds,
\]
From Lemma 2.4, we have
\[
G(t, s) \geq a_2 s(1-s), \quad (t, s) \in \left[ \frac{\eta}{\alpha}, \eta \right] \times [0,1].
\]
Thus,
\[
\min_{t \in \left[ \frac{\alpha}{\eta}, \eta \right]} (Au)(t) \geq \frac{a_2}{a_1} a_1 \int_0^1 s(1-s)f\left(s, \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau\right) ds
\]
\[
\geq \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha - 1, 1\} \|Au\|
\]
\[
= \gamma \|Au\|.
\]
So \( AK \subset K \). \(\square\)

**3. Main Results**

**Proof of Theorem 1.1.** By \((H_2)\), there exist \( c_1 > 0, \varepsilon_1 > 0, \delta \in (0,1) \) such that
\[
c_1 \varepsilon_1 a_1^{r_1+1} a b^{r_1} \leq 1, \quad \varepsilon_1 a_1 b \leq 1, \tag{3.1}
\]
and
\[
q_1(x) \leq c_1 x^{r_1}, \quad q_2(y) \leq \varepsilon_1 y^{r_2}, \quad x \in [0,1], y \in [0,\delta], \tag{3.2}
\]
It follows from (H$_1$) and Lemma 2.3 that for any $u \in \partial B_{\delta} \cap K$,
\[
\int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau \leq \varepsilon_1 \int_0^1 G(s, \tau)p_2(\tau)u^{r_2}(\tau)d\tau
\leq \varepsilon_1 a_1 \int_0^1 \tau(1-\tau)p_2(\tau)d\tau \|u\|^{r_2} = \varepsilon_1 a_1 b \delta^{r_2} \leq 1. \tag{3.3}
\]
Thus, by (H$_1$), (3.1) – (3.3), we have for any $u \in \partial B_{\delta} \cap K, \; t \in [0, 1]$,
\[
(Au)(t) \leq \int_0^1 G(t, s)p_1(s)q_1 \left( \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau \right) d\tau
\leq c_1 \int_0^1 G(t, s)p_1(s) \left( \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau \right)^{r_1} d\tau
\leq c_1 \int_0^1 G(t, s)p_1(s) \left( \varepsilon_1 \int_0^1 G(s, \tau)p_2(\tau)u^{r_2}(\tau)d\tau \right)^{r_1} d\tau
\leq c_1 a_1 \int_0^1 s(1-s)p_1(s)ds \left[ \varepsilon_1 a_1 \int_0^1 \tau(1-\tau)p_2(\tau)d\tau \right]^{r_1} \|u\|^{r_1 r_2}
= c_1 \varepsilon_1 a_1^{r_1+1} b \|u\|^{r_1 r_2} \leq \|u\|^{r_1 r_2} \leq \|u\|
\]
Consequently,
\[
\|Au\| \leq \|u\|, \; \forall u \in \partial B_{\delta} \cap K. \tag{3.4}
\]
On the other hand, by condition (H$_3$), we know that there exist $c_2 > 0$, $\varepsilon_2 > 0$ and $R_1 > 1$ such that
\[
f(t, x) \geq \varepsilon_2 x^{l_1}, \; g(t, y) \geq c_2 y^{l_2}, \; x, y > R_1, t \in [\frac{\eta}{\alpha}, \eta], \tag{3.5}
\]
and $c_2, \varepsilon_2$ satisfy
\[
a_2 c_2 \gamma^{l_2} \int_{\frac{\eta}{\alpha}}^\eta \tau(1-\tau)d\tau \geq 1, \quad \varepsilon_2 c_2^{l_1} a_2^{l_1+1} \gamma^{l_1 l_2} \left( \int_{\frac{\eta}{\alpha}}^\eta \tau(1-\tau)d\tau \right)^{l_1+1} \geq 1. \tag{3.6}
\]
Choose $R > \max \left\{ \frac{R_1}{\gamma}, R_1^{\frac{1}{l_2}} \right\}$. Then for any $u \in \partial B_{R} \cap K$, we have
\[
\min_{t \in [\frac{\eta}{\alpha}, \eta]} u(t) \geq \gamma \|u\| = \gamma R > R_1.
\]
Therefore, by virtue of (3.5) and (3.6), we get that for any $s \in [\frac{\eta}{\alpha}, \eta]$
\[
\int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau \geq \int_{\frac{\eta}{\alpha}}^\eta G(s, \tau)g(\tau, u(\tau))d\tau
\]
Then for any \( t \in [\frac{\eta}{\alpha}, \eta] \) and \( u \in \partial B_R \cap K \), we have
\[
(Au)(t) = \int_0^1 G(t, s) f \left( s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) ds
\geq \int_{\frac{\eta}{\alpha}}^\eta G(t, s) \left( \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right) l_1^{i_1} ds
\geq \varepsilon_2 \int_{\frac{\eta}{\alpha}}^\eta G(t, s) \left( \int_0^1 \tau (1 - \tau) u_l^2(\tau) d\tau \right)^{l_1} l_1
\geq \varepsilon_2 c_2 a_2 c_1 \int_{\frac{\eta}{\alpha}}^\eta \tau (1 - \tau) u_l^2(\tau) d\tau
\geq \frac{c_3 a_1}{\alpha^{\alpha_1}} \int_{\frac{\eta}{\alpha}}^\eta G(t, s) ds
\geq \|u\| l_1^{l_1} \geq \|u\|.
\]

Consequently,
\[
\|Au\| \geq \|u\|, \quad \forall u \in \partial B_R \cap K. \quad (3.7)
\]

By Lemma 2.1, (3.4) and (3.7), we obtain that \( A \) has a fixed point in \((\overline{B_R \setminus B_d}) \cap K\). Therefore, BVP (1.1) has at least one positive solution \((u, v) \in K \times K\) satisfying \( u(t) > 0, v(t) > 0 \) for \( t \in (0, 1) \).

**Proof of Theorem 1.2.** By (H_4), there exist \( c_3 > 0, \varepsilon_3 > 0, N_1 > 0, N_2 > 0 \) such that
\[
q_1(x) \leq c_3 x^{\alpha_1} + N_1, \quad q_2(y) \leq \varepsilon_3 y^{\alpha_2} + N_2, \quad x, y \in \mathbb{R}^+, \quad (3.8)
\]
and satisfying
\[
(2\varepsilon_3)^{\alpha_1} c_3 a_1^{\alpha_1 + 1} ab^{\alpha_1} < 1. \quad (3.9)
\]

Therefore, by (H_1), (3.8), (3.9) and Lemma 2.3, we have
\[(Au)(t) \leq \int_0^1 G(t,s)p_1(s)q_1 \left( \int_0^1 G(s,\tau)g(\tau, u(\tau))d\tau \right) ds \leq \int_0^1 G(t,s)p_1(s) \left[ c_3 \left( \int_0^1 G(s,\tau)g(\tau, u(\tau))d\tau \right)^\alpha_1 + N_1 \right] ds \]

\[= N_1 \int_0^1 G(t,s)p_1(s)ds + c_3 \int_0^1 G(t,s)p_1(s) \left( \int_0^1 G(s,\tau)g(\tau, u(\tau))d\tau \right)^\alpha_1 ds \leq N_1 a_1 a + c_3 a_1^{\alpha_1+1} \int_0^1 s(1-s)p_1(s) \left( \int_0^1 G(s,\tau)p_2(\tau)(\varepsilon_3 u^{\alpha_2}(\tau) + N_2) d\tau \right)^\alpha_1 ds \leq N_1 a_1 a + c_3 a_1^{\alpha_1+1} b^{\alpha_1} \int_0^1 s(1-s)p_1(s)\varepsilon_3 \| u \|^{\alpha_1 \alpha_2} + N_2 \| u \|^{\alpha_1 \alpha_2} \]

\[= N_1 a_1 a + (2N_2)^{\alpha_1} c_3 a_1^{\alpha_1+1} b^{\alpha_1} + (2\varepsilon_3)^{\alpha_1} c_3 a_1^{\alpha_1+1} b^{\alpha_1} \| u \|^{\alpha_1 \alpha_2}. \]

By (3.9), we can choose \( R_2 > 0 \), which is sufficiently large such that

\[
\| Au \| \leq \| u \|, \quad \forall u \in \partial B_{R_2} \cap K.
\] (3.10)

On the other hand, by (H_5), we know that there exist \( \varepsilon_4 > 0, c_4 > 0 \) and \( \rho \in (0,1) \) such that

\[
f(t,x) \geq \varepsilon_4 x^{\beta_1}, \quad g(t,y) \geq c_4 y^{\beta_2}, \quad x,y \in [0,\rho], \quad t \in [\frac{\eta}{\alpha}, \eta],
\] (3.11)

and \( c_4, \varepsilon_4, \rho \) satisfy

\[
\varepsilon_4 c_4^{\beta_1 \gamma_1 \beta_2} a_2^{\beta_1+1} \left( \int_0^\eta s(1-s)ds \right)^{\beta_1+1} \geq 1.
\] (3.12)

Since \( q_2(0) = 0 \) and the continuity of \( q_2(u) \), there exists \( \varepsilon \in (0,\rho) \), which is sufficiently small such that

\[
q_2(u) \leq a_1^{-1} b^{-1} \rho, \quad \forall u \in [0,\varepsilon].
\]

Thus, for any \( u \in \partial B_\varepsilon \cap K \) and \( s \in [0,1] \), we have

\[
\int_0^1 G(s,\tau)g(\tau, u(\tau))d\tau \leq \int_0^1 G(s,\tau)p_2(\tau)q_2(\tau) d\tau \leq a_1 \int_0^1 (1-\tau)p_2(\tau) a_1^{-1} b^{-1} \rho d\tau = \rho.
\] (3.13)
By Lemma 2.4, (3.11) and (3.13), we know that for $t \in \left[\frac{\eta}{\alpha}, \eta\right]$

$$(Au)(t) \geq \varepsilon_4 \int_{\frac{\eta}{\alpha}}^{\eta} G(t, s) \left( \int_{0}^{1} G(s, \tau)g(\tau, u(\tau))d\tau \right)^{\beta_1} ds$$

$$\geq \varepsilon_4 \int_{\frac{\eta}{\alpha}}^{\eta} G(t, s) \left( c_4 \int_{0}^{1} G(s, \tau)u^{\beta_2}(\tau)d\tau \right)^{\beta_1} ds$$

$$\geq \varepsilon_4 a_2 \int_{\frac{\eta}{\alpha}}^{\eta} s(1-s) \left[ (a_2c_4)^{\beta_1} \left( \int_{\frac{\eta}{\alpha}}^{\eta} \tau(1-\tau)u^{\beta_2}(\tau)d\tau \right)^{\beta_1} \right] ds$$

$$\geq \varepsilon_4 c_4^{\beta_1} \gamma_1^{\beta_1} \beta_2 a_2^{\beta_1+1} \left( \int_{\frac{\eta}{\alpha}}^{\eta} s(1-s)ds \right)^{\beta_1+1} \|u\|^{\beta_1 \beta_2}$$

$$\geq \|u\|^{\beta_1 \beta_2} \geq \|u\|.$$ 

Consequently,

$$\|Au\| \geq \|u\|, \quad \forall u \in \partial B_\varepsilon \cap K. \quad (3.14)$$

By Lemma 2.1, (3.10) and (3.14), $A$ has at least one fixed point in $(\overline{B}_{R_2} \setminus B_\varepsilon) \cap K$. Therefore, BVP (1.1) has at least one positive solution $(u, v) \in K \times K$ satisfying $u(t) > 0, v(t) > 0$ for $t \in (0, 1)$. □

**Proof of Theorem 1.3.** By (H_6), for any $u \in \partial B_N \cap K$ and $s \in [0, 1]$, we get

$$\int_{0}^{1} G(s, \tau)g(\tau, u(\tau))d\tau \leq \int_{0}^{1} G(s, \tau)p_2(\tau)q_2(u(\tau))d\tau \leq Ma_1 \int_{0}^{1} \tau(1-\tau)p_2(\tau)d\tau = Ma_1 b.$$ 

Thus, for any $u \in \partial B_N \cap K$ and $t \in [0, 1]$, we get

$$(Au)(t) \leq \int_{0}^{1} G(t, s)p_1(s)q_1 \left( \int_{0}^{1} G(s, \tau)g(\tau, u(\tau))d\tau \right) ds$$

$$\leq \sup_{u \in [0, Ma_1 b]} q_1(u) a_1 \int_{0}^{1} s(1-s)p_1(s)ds$$

$$= a_1 a \cdot \sup_{u \in [0, Ma_1 b]} q_1(u) \leq N.$$ 

Consequently

$$\|Au\| \leq \|u\|, \quad \forall u \in \partial B_N \cap K, \quad (3.15)$$

where $M = \sup_{u \in [0, N]} q_2(u)$, $b$ is the same as in (H_1).

On the other hand, by (H_3) and (H_5), for sufficiently large $R > N$ and sufficiently small $\varepsilon \in (0, N)$, (3.7) and (3.14) hold. Thence, by (3.7), (3.14)
and (3.15), we obtain $A$ has at least one fixed point in $(\overline{B}_R \setminus B_N) \cap K$ and $(\overline{B}_N \setminus B_{\varepsilon}) \cap K$ respectively. Therefore, BVP (1.1) has at least two positive solutions $(u_1, v_1) \in K \times K$, $(u_2, v_2) \in K \times K$ satisfying $u_i(t) > 0, v_i(t) > 0$ for $t \in (0,1) \ (i = 1, 2)$. □

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References


