DENSITY OF $C_0^\infty(\mathbb{R}^n)$ IN $W^{1,p(x)}(\mathbb{R}^n)$
WITH ASSUMPTION DENSITY OF $C^1(\mathbb{R}^n)$

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Abstract: In this paper, we present a sufficient condition for the density of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p(x)}(\mathbb{R}^n)$ with assumption that $p(x)$ satisfying a condition such that $C^1(\mathbb{R}^n)$ is dense in $W^{1,p(x)}(\mathbb{R}^n)$. The origin of our work comes from a similar question of Hästö [5] under the density of continuous or Hölder continuous functions.

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1. Introduction

Variable exponent analysis has become a growing field of interest since Kováčik and Rákosník paper [6]. Many basic properties of Lebesgue and Sobolev spaces were shown in their paper. Variable Lebesgue spaces are a generalization of Lebesgue spaces where we allow the exponent to be a measurable function and thus the exponent may vary, and has found numerous important applications with Sobolev spaces. Examples are fluid dynamics, elasticity theory, differential equations with non-standard growth conditions and image restoration (cf. [2,
8, 7]). On the basic properties of the variable exponent Lebesgue and Sobolev spaces we refer to [3, 6].

Let \( p : \Omega \to [1, \infty) \) be a measurable bounded function, called a variable exponent on \( \Omega \). We also define \( p^+ = \text{esssup}_{x \in \mathbb{R}^n} p(x) \) and \( p^- = \text{essinf}_{x \in \mathbb{R}^n} p(x) \).

The class \( C_0^\infty(\mathbb{R}^n) \) of infinitely differentiable functions with compact support in \( \Omega \) is dense in the spaces \( L^{p(.)}(\Omega) \), which was established among the first basic properties of these spaces in [6].

Variable exponent Lebesgue spaces do not have the mean continuity property. If \( p \) is continuous and non-constant function in an open ball \( B \), then there exists a function \( L^{p(.)}(\Omega) \) such that \( u(x + h) \notin L^{p(.)}(\Omega) \) for \( h \in \mathbb{R}^n \) with arbitrary small norm.

We define the variable exponent Lebesgue space \( L^{p(.)}(\Omega) \) to consist of all measurable functions \( u : \Omega \to \mathbb{R} \) for which the modular \( \varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx \) is finite. We define the Luxemburg norm on this space by

\[
\|u\|_{L^{p(.)}(\Omega)} = \|u\|_{p(.)} = \inf\{\lambda > 0 : \varrho_{p(.)}(u/\lambda) \leq 1\}.
\]

If \( p \) is a constant function, then the variable exponent Lebesgue spaces coincides with the classical Lebesgue space. One central property of these spaces reads: If \( p^+ < \infty \) and \( (u_i) \) is a sequence of functions in \( L^p \), then \( \|u_i\|_{p(.)} \to 0 \) if and only if \( \varrho_{p(.)}(u_i) \to 0 \). This and many other basic results were proven in [6].

The variable exponent Sobolev space \( W^{1,p(.)}(\Omega) \) is the subspace of functions \( u \in L^{p(.)}(\Omega) \) whose distributional gradient exists almost everywhere and satisfies \( |\nabla u| \in L^{p(.)}(\Omega) \). The norm \( \|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)} \) makes \( W^{1,p(.)}(\Omega) \) a Banach space. We also define a modular in the Sobolev space by \( \varrho_{1,p(.)}(u) = \varrho_{p(.)}(u) + \varrho_{p(.)}(|\nabla u|) \).

It is known that in general, \( C^\infty(\Omega) \) may not be dense in \( W^{1,p(.)}(\Omega) \). The first example is given by Zhikov [10, 11] as follows:

\[
\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\},
\]

\[
p(x) = \begin{cases} 
\alpha_1 & \text{if } x_1 x_2 > 0, \\
\alpha_2 & \text{if } x_1 x_2 < 0,
\end{cases}
\]

where \( 1 < \alpha_1 < 2 < \alpha_2 \), then \( C^\infty(\Omega) \) is not dense in \( W^{1,p(.)}(\Omega) \).

Edmunds and Rakosnik [4] proved denseness under some special monotonicity type condition on \( p(x) \). In [1, 9] another type of sufficient condition to ensure the density of \( C^\infty(\Omega) \) in \( W^{1,p(.)}(\Omega) \) is the so-called logarithmic Holder continuity, which is expressed as

\[
|p(x) - p(y)| \leq \frac{C}{-\ln|x - y|}, \quad \text{for all } x, y \in \Omega, \quad |x - y| \leq \frac{1}{2}.
\]
2. Density

After the Zhikov nondensity example it became more significant to look at less smoothness density of functions in variable exponent Sobolev spaces. Hasto [5] made some work in this direction and asked the following.

**Question 2.1.** Suppose that $C(\Omega)$ or $C^\alpha(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. Is it then true that smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$?

Since the derivative of a Sobolev function may be unbounded it seems for convolution the assumption of density of continuous functions does not help. Similarly, we assume density of $C^1(\mathbb{R}^n)$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ instead of density $C(\Omega)$ and show the following theorem.

**Theorem 2.2.** Let $p(\cdot) \in P(\mathbb{R}^n)$ be a bounded variable exponent. Suppose that $C^1(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, then $C^\infty_0(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.

**Proof.** We first show that $C^1_0(\mathbb{R}^n)$ is dense in $C^1(\mathbb{R}^n)$ on the norm of $W^{1,p(\cdot)}(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n)$ and $\eta \in C^1_0(\mathbb{R}^n)$ be a compactly supported function which equals 1 near the origin, and consider the functions $u_R(x) = u(x)\eta(\frac{x}{R})$ for $R > 0$. It is easy to see $u_R \in C^1_0(\mathbb{R}^n)$. Since $p^+ < \infty$, instead of $\|u_R - u\|_{p(\cdot)} \to 0$ norm convergence, by using Lebesgue dominated convergence in the modular we see that

$$\int_{\mathbb{R}^n} |u_R - u|^{p(x)} \, dx \to 0$$

holds as $R \to \infty$.

We now show $\|((\nabla u_R)(x) - (\nabla u)(x))\|_{p(\cdot)} \to 0$ as $R \to \infty$. By product rule we get

$$\nabla u_R = (\nabla u)(x)\eta(\frac{x}{R}) + \frac{1}{R} u(x)(\nabla \eta)(\frac{x}{R}).$$

Let us prove the convergence:

$$\|(\nabla u_R)(x) - (\nabla u)(x)\|_{p(\cdot)} = \|(\nabla u)(x)\eta(\frac{x}{R}) + \frac{1}{R} u(x)(\nabla \eta)(\frac{x}{R}) - (\nabla u)(x)\|_{p(\cdot)}$$

$$\leq \|(\nabla u)(x)\eta(\frac{x}{R}) - (\nabla u)(x)\|_{p(\cdot)} + \frac{1}{R} \|u(x)(\nabla \eta)(\frac{x}{R})\|_{p(\cdot)}.$$

Again since $p^+ < \infty$, we use modular convergence for the first term and by using Lebesgue dominated convergence the first term goes to zero as $R \to \infty$ and it is obvious the second term goes to zero as $R \to \infty$.

We now show that $C^\infty_0(\mathbb{R}^n)$ is dense in $\in C^1(\mathbb{R}^n)$. 
Since $u \in C^1_0(\mathbb{R}^n)$ has compact support, we denote this set by spt $u = K$ and from $K$ we define larger compact set as follows:

$$K_{\delta(\epsilon)} = \{ x \in \mathbb{R}^n : \text{dist}(x, K) \leq \delta(\epsilon) \},$$

where $\delta(\epsilon)$ will be chosen later and will be taken such that $0 < \delta(\epsilon) < 1$.

With this condition $\nabla u$ and $u$ are uniformly continuous on $K_{\delta(\epsilon)}$. Let $0 < \epsilon < 1$, then we get

$$|u(x - y) - u(x)| < \epsilon \quad \text{as} \quad |y| < \delta_1(\epsilon)$$

and

$$|\nabla u(x - y) - \nabla u(x)| < \epsilon \quad \text{as} \quad |y| < \delta_1(\epsilon)$$

in which $\delta_1(\epsilon), \delta_2(\epsilon) < \epsilon$ can be chosen. We get

$$\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\},$$

as we promised.

Let $\phi_{\delta(\epsilon)}$ be a standard mollifier. Then

$$\|u * \phi_{\delta(\epsilon)} - u\|_{W^{1,p}(\mathbb{R}^n)} = \|u * \phi_{\delta(\epsilon)} - u\|_{p(.)} + \|\nabla u * \phi_{\delta(\epsilon)} - \nabla u\|_{p(.)}.$$  

Since $p^+ < \infty$, we show convergence in the modular:

$$\int_{\mathbb{R}^n} \left| \int_{B(0, \delta(\epsilon))} |u(x - y) - u(x)| \phi_{\delta(\epsilon)}(y) \, dy \right|^{p(x)} \, dx \leq e^{p^-} |K_{\delta(\epsilon)}|$$

and

$$\int_{\mathbb{R}^n} \left| \int_{B(0, \delta(\epsilon))} |\nabla u(x - y) - \nabla u(x)| \phi_{\delta(\epsilon)}(y) \, dy \right|^{p(x)} \, dx \leq e^{p^-} |K_{\delta(\epsilon)}|$$

are obtained. Since $|K_{\delta(\epsilon)}| < \infty$ the convergence holds.

Thus we show that $C^1_0(\mathbb{R}^n)$ is dense in $C^1(\mathbb{R}^n)$ and $C^\infty_0(\mathbb{R}^n)$ is dense $C^1_0(\mathbb{R}^n)$.

By the assumption of theorem, the density that we have showed and triangular inequality of norm easily show that $C^\infty_0(\mathbb{R}^n)$ is also dense in $W^{1,p}(\mathbb{R}^n)$.
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References


