A GENERALIZATION OF CONVEX FUNCTIONS

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Abstract: The object of this paper is to obtain sharp results involving growth and distortion properties for the classes $V_k$ and $C^*_k$ of analytic functions in the unit disk.

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1. Introduction and Definitions

Let $S$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{ z : |z| < 1 \}$.

Let $C$ denote the class of convex functions:

$$f(z) \in C \text{ if and only if for } z \in E, \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0.$$  

For $k \geq 2$, denote by $V_k$ the class of normalized functions of bounded boundary rotation at most $k\pi$. Thus $g(z) \in V_k$ if and only if $g(z)$ is analytic in $E$, $g'(z) \neq 0$, $g(0) = g'(0) - 1 = 0$ and for $z \in E$:

$$\int_0^{2\pi} \left| \Re \frac{zg'(z)}{g(z)} \right| d\theta \leq k\pi. \quad (1.1)$$
It is known [1] that for \(2 \leq k \leq 4\), \(V_k\) consists only of univalent functions (in fact, close-to-convex functions). This is another definition of the class \(V_k\).

For fixed \(k\), let \(V_k\) denote the class of functions \(g(z)\) normalized so that

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]

which are analytic in \(E\) and have an integral representation of the form

\[
g'(z) = \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \log (1 - z e^{-it})^{-1} d\mu(t) \right\}, \tag{1.2}
\]

where \(\mu(t)\) is real-valued and bounded variation on \([0, 2\pi]\) with

\[
\int_0^{2\pi} d\mu(t) = 2\pi, \tag{1.3}
\]

\[
\int_0^{2\pi} |d\mu(t)| \leq k\pi.
\]

The representation formula (1.2) together with (1.3) is due to Paatero [1] and is equivalent to the definition (1.1) for \(g(z) \in V_k\).

In 1917, Lowner [2] was the first to consider functions of bounded boundary rotation. Later, Paatero [1, 3] made an exhaustive study of the class. The function \(g_k(z)\) defined for \(z \in E\) by

\[
g_k(z) = \frac{1}{k} \left[ \left( \frac{1+z}{1-z} \right)^{\frac{1}{k}} - 1 \right] = \sum_{n=1}^{\infty} B_n(k) z^n \tag{1.4}
\]

belongs to \(V_k\) and is extremal for many problems.

Paatero [1] proved sharp distortion theorems for \(g(z) \in V_k\).

Let \(f(z)\) be analytic in \(E\), \(f'(0) \neq 0\) and normalized so that

\[
f(0) = 0, \quad f'(0) = 1.
\]

Then, for \(k \geq 2\), \(f(z) \in T_k\) if there exists a function \(g(z) \in V_k\), such that for \(z \in E\)

\[
\Re \frac{f'(z)}{g'(z)} > 0. \tag{1.5}
\]

Clearly \(T_2 = K\), the class of close-to-convex functions.

We now define a new subclass \(C_k^*\) which has the same relationship with \(T_k\) as \(C\) has with \(S^*\) (the class of starlike functions).
Let \( f(z) \) be analytic in \( E \) and normalized so that \( f(0) = 0, f'(0) = 1 \) and \( f'(z) \neq 0 \). Then \( f(z) \in C^*_k \) \((k \geq 2)\) if there exists a function \( g(z) \in V_k \) such that for \( z \in E \)

\[
\Re \left( \frac{zf'(z)}{g'(z)} \right) > 0. \tag{1.6}
\]

Clearly, \( C_2^* = C^* \), the class of quasi-convex functions [5]. It follows easily from definition (1.6) that

\( f(z) \in C^*_k \) if and only if \( zf'(z) \in T_k \). \tag{1.7}

Let us consider the function

\[
F_k(z) = \frac{1}{k+2} \left[ \left( \frac{1+z}{1-z} \right)^{\frac{k}{2}+1} - 1 \right] = z + \sum_{n=2}^{\infty} A_n(k)z^n. \tag{1.8}
\]

It is then easy to show that [7], \( F_k(z) \in T_k \).

### 2. Known Results

**Theorem 2.1.** (see [1]) If \( g(z) \in V_k \), then for \( z = re^{i\theta} \in E \)

\[
\frac{1}{k} \left[ 1 - \left( \frac{1-r}{1+r} \right)^{\frac{k}{2}} \right] \leq |g(z)| \leq \frac{1}{k} \left[ \left( \frac{1+r}{1-r} \right)^{\frac{k}{2}} - 1 \right]
\]

\[
\frac{1}{1-r^2} \left( \frac{1-r}{1+r} \right)^{\frac{k}{2}} \leq g'(z) \leq \frac{1}{1-r^2} \left( \frac{1+r}{1-r} \right)^{\frac{k}{2}}.
\]

The function \( g_k(z) \) defined for \( z \in E \) by

\[
g_k(z) = \frac{1}{k} \left[ \left( \frac{1+z}{1-z} \right)^{\frac{k}{2}} - 1 \right] = \sum_{n=2}^{\infty} B_n z^n
\]

shows that these inequalities are sharp.

**Theorem 2.2.** (see [6]) Let \( g(z) \in V_k \) and \( \xi \in E \). Then \( F(z) \in V_k \), where \( F(z) \) is given by

\[
F(z) = \frac{g \left( \frac{\xi+z}{1+\xi z} \right) - g(\xi)}{g'(\xi) \left[ 1 - |\xi|^2 \right]}.
\]
Theorem 2.3. (see [7]) Let \( f(z) \in T_k \), then for \( z = re^{i\theta} \in E \)

\[
\frac{(1 - r)^{\frac{k}{2}}}{(1 + r)^{\frac{k}{2} + 2}} \leq |f'(z)| \leq \frac{(1 + r)^{\frac{k}{2}}}{(1 - r)^{\frac{k}{2} + 2}}.
\]

These bounds are sharp, equality being attained for the functions \( F_k(z) \) defined by

\[
F_k(z) = \frac{1}{k + 2} \left[ \left( \frac{1 + z}{1 - z} \right)^{\frac{k}{2} + 1} - 1 \right].
\]

3. Some of the Basic Properties of Functions in the Classes \( V_k \) and \( C_k^* \)

Theorem 3.1. Let \( g(z) \in V_k \), then for \( z = re^{i\theta} \in E \)

\[
\frac{k(1 - r)^{\frac{k-1}{2}}}{(1 + r)^{\frac{k+1}{2} + 1} \left[ 1 - \left( \frac{1-r}{1+r} \right)^{\frac{k}{2}} \right]} \leq \frac{|g'(z)|}{|g(z)|} \leq \frac{k(1 + r)^{\frac{k-1}{2}}}{(1 - r)^{\frac{k+1}{2} + 1} \left[ (1 + \frac{1}{1-r})^{\frac{k}{2}} - 1 \right]}.
\]

The result is sharp.

Proof. Since \( g(z) \in V_k \), Theorem 2.2 shows that \( F(z) \) defined by

\[
F(z) = \frac{g(\frac{z + \xi}{1 + \xi z}) - g(\xi)}{g'(\xi) [1 - |\xi|^2]}
\]

is in \( V_k \), when \( \xi \) is any arbitrary point in \( E \). Thus, with \( z = -\xi \) we obtain

\[
F(-\xi) = \frac{-g(\xi)}{g'(\xi) [1 - |\xi|^2]},
\]

(3.2)

Now, using the distortion Theorem 2.1 for \( g(z) \in V_k \), we obtain

\[
\frac{1}{k} \left[ 1 - \left( \frac{1 - |\xi|}{1 + |\xi|} \right)^{\frac{k}{2}} \right] \leq |F(-\xi)| \leq \frac{1}{k} \left[ \left( \frac{1 + |\xi|}{1 - |\xi|} \right)^{\frac{k}{2}} - 1 \right]
\]

and so (3.2) gives

\[
\frac{1}{k} \left[ 1 - \left( \frac{1 - |\xi|}{1 + |\xi|} \right)^{\frac{k}{2}} \right] \leq \frac{-g(\xi)}{g'(\xi)(1 - |\xi|^2)} \leq \frac{1}{k} \left[ \left( \frac{1 + |\xi|}{1 - |\xi|} \right)^{\frac{k}{2}} - 1 \right],
\]
that is
\[
(1 - |\xi|^2) \frac{1}{k} \left[ 1 - \left( \frac{1 - |\xi|}{1 + |\xi|} \right)^{\frac{k}{2}} \right] \leq \left| \frac{g'(\xi)}{g(\xi)} \right| \leq \frac{1}{k} \left( \frac{1 + |\xi|}{1 - |\xi|} \right)^{\frac{k}{2}} - 1 \left(1 - |\xi|^2\right).
\]
Since \(\xi\) is an arbitrary point in \(E\), the result follows.

**Remark.** We note that when \(k = 2\) we obtain the classical distortion theorem for the class \(C\) of normalized convex functions.

**Theorem 3.2.** Let \(f(z) \in C_k\), then for \(z \in \mathbb{R} e^{i\theta} \in E\) and \(k \geq 2\):
\[
\frac{1}{k + 2} \left[ 1 - \left( \frac{1 - r}{1 + r} \right)^{\frac{k+1}{2}} \right] \leq |zf'(z)| \leq \frac{1}{k + 2} \left[ \left( \frac{1 + r}{1 - r} \right)^{\frac{k+1}{2}} - 1 \right],
\]
\[
\frac{1}{k + 2} \int_0^r \left[ 1 - \left( \frac{1 - t}{1 + t} \right)^{\frac{k+1}{2}} \right] \frac{dt}{t} \leq |f(z)| \leq \frac{1}{k + 2} \int_0^r \left[ \left( \frac{1 + t}{1 - t} \right)^{\frac{k+1}{2}} - 1 \right] \frac{dt}{t}.
\]

**Proof.** Since \(f(z) \in C_k^*\), (1.7) shows that \(zf'(z) \in T_k\). Thus, from Theorem 2.3,
\[
\frac{1}{k + 2} \left[ 1 - \left( \frac{1 - r}{1 + r} \right)^{\frac{k+1}{2}} \right] \leq |zf'(z)| \leq \frac{1}{k + 2} \left[ \left( \frac{1 + r}{1 - r} \right)^{\frac{k+1}{2}} - 1 \right]. \tag{3.3}
\]
Integrating the right-hand inequality in (3.3) from 0 to \(z\), we obtain
\[
|f(z)| \leq \int_0^{|z|} |f'(z)| |dz| \leq \frac{1}{k + 2} \int_0^r \left[ \left( \frac{1 + t}{1 - t} \right)^{\frac{k+1}{2}} - 1 \right] \frac{dt}{t}.
\]
In order to obtain a lower bound for \(|f(z)|\), we proceed as follows. Let \(z_1\) be such that \(|z_1| = r\) and \(|f(z_1)| \leq |f(z)|\) for all \(z\) with \(|z| = r\). Writing \(\omega = f(z)\), it follows that the line-segment \(\lambda\) from \(\omega = 0\) to \(\omega = f(z)\) lies entirely in the image of \(f(z)\). Let \(L\) be the pre-image of \(\lambda\). Then
\[
|f(z)| \geq |f(z_1)| = \int_{\lambda} |d\omega| = \int_{\Omega} \left| \frac{d\omega}{dz} \right| |dz| \geq \frac{1}{k + 2} \int_0^r \left[ 1 - \left( \frac{1 - t}{1 + t} \right)^{\frac{k+1}{2}} \right] \frac{dt}{t}.
\]
Then, the theorem follows. The function \(\Phi\) defined by
\[
\Phi(z) = \frac{1}{k + 2} \int_0^z \left[ \left( \frac{1 + t}{1 - t} \right)^{\frac{k+1}{2}} - 1 \right] \frac{dt}{t}
\]
shows that equality can occur.
References


