SECOND HANKEL DETERMINANT FOR
CERTAIN CLASSES OF ANALYTIC FUNCTIONS

C. Selvaraj¹, T.R.K. Kumar² §

¹Presidency College
Chennai, 600 005, Tamilnadu, INDIA
²R.M.K. Engineering College
R.S.M. Nagar, Kavaraipettai, 601 206, Tamilnadu, INDIA

Abstract: In the present investigation, we introduce some new subclasses of analytic-univalent functions and determine the sharp upper bounds of the second Hankel determinant for the functions belonging to such classes.

AMS Subject Classification: 30C45
Key Words: analytic functions, starlike functions, convex functions, close to convex functions, starlike functions with respect to symmetric points, close-to-convex functions with respect to symmetric points, Hankel determinant

1. Introduction, Definitions and Preliminaries

We let $\mathcal{A}$ to denote the class of functions analytic in $\mathbb{U}$ and having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1)

in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of functions $f(z) \in \mathcal{A}$ and univalent in $\mathbb{U}$.
The $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [27, 28] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$ 

This determinant has been considered by several authors in the literature, see [24]. For example, Noor [25] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $U$ with bounded boundary. Later, Ehrenborg [7] considered the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some its properties were discussed by thoroughly by Layman in [15].

Also, the Hankel determinant was studied by various authors including Hayman [12] and Pommerenke [29]. Easily, one can observe that the Fekete-Szegő functional is $H_2(1)$. Then Fekete-Szegő further generalized the estimate $|a_3 - \mu a_2^2|$, where $\mu$ is real and $f \in U$. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő functional. For the discussion in this paper, the Hankel determinant for the case $q = 2$ and $n = 2$ are being considered

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$ 

Janteng, Halim and Darus [14] have determined the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the functions $f$ in the subclass $\mathcal{RT}$ of $U$, consisting of functions whose derivative has a positive real part studied by Mac Gregor [18]. In their work, they have shown that if $f \in \mathcal{RT}$ then $|a_2a_4 - a_3^2| \leq \frac{1}{4}$. The same authors [12] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex denoted by $\mathcal{ST}$ and $\mathcal{CV}$ of $U$ and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$, respectively. Mishra and Gochhayat [21] have obtained sharp bound to the non-linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)$ ($0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2}$).

Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors, see e.g. [1], [3], [4], [9-11], [21], [22], [29], [31-39].

Motivated by the above mentioned results obtained by different authors in this direction, we seek upper bound of the function $|a_3 - \mu a_2^2|$ for functions belonging to the defined classes.
A function $f(z) \in \mathcal{A}$ is said to be in the class $RST(\alpha) (\alpha \geq 0)$ in $\mathbb{U}$, if it satisfies the condition:

$$
Re \left[ \alpha f'(z) + (1 - \alpha) \frac{zf'(z)}{f(z)} \right] > 0, \quad \forall z \in \mathbb{U}.
$$  \hspace{1cm} (2)

This class was studied by D. Vamshee Krishna and T. Ramreddy. It is observed that for $\alpha = 0$ and $\alpha = 1$ in (2), we respectively get $RST(0) = ST$ and $RST(1) = RT$.

$C(\alpha)$ denotes the subclass of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$
Re \left[ \alpha f'(z) + (1 - \alpha) \frac{zf'(z)}{g(z)} \right] > 0,
$$  \hspace{1cm} (3)

where

$$
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*.
$$  \hspace{1cm} (4)

In particular,

1. $C(1) \equiv RT$,
2. $C(0) \equiv CC$, the class of close-to-convex functions.

Let $C'_*(\alpha)$ be the subclass of functions $f(z) \in \mathcal{A}$, satisfying the condition

$$
Re \left[ \alpha f'(z) + (1 - \alpha) \frac{zf'(z)}{h(z)} \right] > 0,
$$  \hspace{1cm} (5)

where

$$
h(z) = z + \sum_{n=2}^{\infty} d_n z^n \in K.
$$  \hspace{1cm} (6)

We have the following obvious observations:

1. $C'_*(1) \equiv RT$,
2. $C'_*(0) \equiv C'$.

$C_s^{*(\alpha)}$ denote the subclass of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$
Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf'(z)}{f(z) - f(-z)} \right) \right] > 0.
$$  \hspace{1cm} (7)

The following observations are obvious:
1. $C_{s}^{*(1)} \equiv RT$,

2. $C_{s}^{*(0)} \equiv S_{s}^{*}$, the class of starlike functions with respect to symmetric points introduced by Sakaguchi [33].

Let $C_{s}^{\alpha}$ be the subclass of functions $f(z) \in A$, satisfying the condition

$$Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf'(z)}{g(z) - g(-z)} \right) \right] > 0,$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_{n}z^{n} \in S_{s}^{*}.$$  

In particular,

1. $C_{s}^{1} \equiv RT$,

2. $C_{s}^{0} \equiv C_{s}$, the class of close-to-convex functions with respect to symmetric points introduced by Das and Singh [6].

Let $C_{1(s)}^{\alpha}$ be the subclass of functions $f(z) \in A$, satisfying the condition

$$Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf'(z)}{h(z) - h(-z)} \right) \right] > 0,$$

where

$$h(z) = z + \sum_{n=2}^{\infty} d_{n}z^{n} \in K_{s}.$$  

We have the following obvious observations:

1. $C_{1(s)}^{(1)} \equiv RT$,

2. $C_{1(s)}^{(0)} \equiv C_{s}'$. 

2. Preliminary Results

Let $P$ be the family of all functions $p$ analytic in $\mathbb{U}$ for which $\text{Re}(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad \forall z \in \mathbb{U}.$$  \hspace{1cm} (12)

**Lemma 2.1.** ([26],[30]) $|p_k| \leq 2, \ (k = 1, 2, 3, \ldots)$.

**Lemma 2.2.** If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2) x,$$

$$4p_3 = p_1^3 + 2p_1 (4 - p_1^2) x - p_1 (4 - p_1^2) x^2 + 2 (4 - p_1^2) \left(1 - |x|^2\right) z,$$

for some $x$ and $z$ satisfying $|x| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewicz [15,16].

3. Main Results

**Theorem 3.1.** If $f(z) \in C(\alpha)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(3 - \alpha)^2}{9}. \hspace{1cm} (13)$$

**Proof.** Since $C_s^{(\alpha)}$ denotes the subclass of functions $f(z) \in A$, satisfying the condition (8), so

$$\alpha f'(z) + (1 - \alpha) \frac{z f''(z)}{g(z)} = p(z), \hspace{1cm} (14)$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*.$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*.$$  \hspace{1cm} (15)
On equating the coefficients of \(z, z^2\) and \(z^3\) in the expansion of (14), we have

\[
\begin{align*}
  a_2 &= \frac{p_1}{2} + \frac{b_2 (1 - \alpha)}{2} \\ 
  a_3 &= \frac{p_2}{2} + \frac{(1 - \alpha) (2a_2 b_2 + b_3 - b_2^2)}{2} \\ 
  a_4 &= \frac{p_3}{4} + \frac{(1 - \alpha) (3a_3 b_2 + 2a_2 b_3 + b_4 - 2a_2 b_2^2 - 2b_2 b_3 + b_2^3)}{4} 
\end{align*}
\]  

(16)

From (15), we can easily verify that

\[
\begin{align*}
  b_2 &= p_1, \\ 
  b_3 &= \frac{p_2 + p_1^2}{2}, \\ 
  b_4 &= \frac{p_3}{3} + \frac{p_1 p_2}{2} + \frac{p_1^3}{6}.
\end{align*}
\]

So (16) yields

\[
\begin{align*}
  a_2 &= \frac{(2 - \alpha) p_1}{2} \\ 
  a_3 &= \frac{(3 - \alpha) p_2}{6} + \frac{(1 - \alpha) (3 - 2\alpha) p_1^2}{6} \\ 
  a_4 &= \frac{(4 - \alpha) p_3}{12} + \frac{(1 - \alpha) (2 - \alpha) p_1 p_2}{4} + \frac{(1 - \alpha) (3\alpha^2 - 6\alpha + 2) p_1^3}{12}
\end{align*}
\]  

(17)

Using (17) yields

\[
a_2a_4 - a_3^2 = \frac{1}{72} \begin{bmatrix}
  3 (2 - \alpha)(4 - \alpha) p_1 p_3 \\
  + 9 (1 - \alpha)(2 - \alpha)^2 - 4 (1 - \alpha)(3 - \alpha)(3 - 2\alpha) p_1^2 p_2 \\
  + 3 (1 - \alpha)(2 - \alpha)(3\alpha^2 - 6\alpha + 2) - 2 (1 - \alpha)^2 (3 - 2\alpha)^2 p_1^4 \\
  - 2 (3 - \alpha)^2 p_2^2
\end{bmatrix}.
\]

(18)

Using Lemma 2.1 and Lemma 2.2 in (18), we have

\[
|a_2a_4 - a_3^2| = \frac{1}{288} \begin{bmatrix}
  3 (2 - \alpha)(4 - \alpha) + 2\alpha^2 (1 - \alpha) \\
  - 4 (1 - \alpha)(\alpha^3 - 4\alpha^2 + 6) - 2 (3 - \alpha)^2
\end{bmatrix} p_1^4
\]
\[-6 (2 - \alpha) (4 - \alpha) - 2\alpha^2 (1 - \alpha) - 4 (3 - \alpha)^2 p_1^2 (4 - p_1^2) x
- \left[ 3 (2 - \alpha) (4 - \alpha) p_1^2 + 2 (3 - \alpha)^2 (4 - p_1^2) \right] (4 - p_1^2) x^2
+ 6 (2 - \alpha) (4 - \alpha) p_1 (4 - p_1^2) \left( 1 - |x|^2 \right) z \].

Assume that $p_1 = p$ and $p \in [0, 2]$, using the triangular inequality and $|z| \leq 1$, we have

\[
|a_2a_4 - a_3^2|
\leq \frac{1}{288} \left[ \left[ 3 (2 - \alpha) (4 - \alpha) + 2\alpha^2 (1 - \alpha) - 4 (1 - \alpha) (\alpha^3 - 4\alpha^2 + 6)
- 2 (3 - \alpha)^2 \right] p^4 + \left[ -6 (2 - \alpha) (4 - \alpha) - 2\alpha^2 (1 - \alpha)
- 4 (3 - \alpha)^2 \right] p^2 (4 - p^2) |x|
+ \left[ 3 (2 - \alpha) (4 - \alpha) p^2 + 2 (3 - \alpha)^2 (4 - p^2) \right] (4 - p^2) |x|^2
+ 6 (2 - \alpha) (4 - \alpha) (4 - p^2) p \left( 1 - |x|^2 \right) \right],
\]

\[
|a_2a_4 - a_3^2|
\leq \frac{1}{288} \left[ \left[ 3 (2 - \alpha) (4 - \alpha) + 2\alpha^2 (1 - \alpha) - 4 (1 - \alpha) (\alpha^3 - 4\alpha^2 + 6)
- 2 (3 - \alpha)^2 \right] p^4 + \left[ -6 (2 - \alpha) (4 - \alpha) - 2\alpha^2 (1 - \alpha)
- 4 (3 - \alpha)^2 \right] p^2 (4 - p^2)
+ \left[ 3 (2 - \alpha) (4 - \alpha) p^2 + 2 (3 - \alpha)^2 (4 - p^2) \right] (4 - p^2) \delta
+ 2 (3 - \alpha)^2 (4 - p^2) \delta^2 \right].
\]

Therefore

\[
|a_2a_4 - a_3^2| \leq \frac{1}{288} F(\delta),
\]

where $\delta = |x| \leq 1$ and

\[
F(\delta) = \left[ 3 (2 - \alpha) (4 - \alpha) + 2\alpha^2 (1 - \alpha) - 4 (1 - \alpha) (\alpha^3 - 4\alpha^2 + 6)
- 2 (3 - \alpha)^2 \right] p^4 + \left[ -6 (2 - \alpha) (4 - \alpha) - 2\alpha^2 (1 - \alpha)
- 4 (3 - \alpha)^2 \right] p^2 (4 - p^2)
+ \left[ 3 (2 - \alpha) (4 - \alpha) p^2 + 2 (3 - \alpha)^2 (4 - p^2) \right] (4 - p^2) \delta
+ 2 (3 - \alpha)^2 (4 - p^2) \delta^2 \right].
\]
\[ + [3 (2 - \alpha) (4 - \alpha) p^2 - 6 (2 - \alpha) (4 - \alpha) p \]
\[ + 2 (3 - \alpha)^2 (4 - p^2) (4 - p^2) \delta^2 ] \]

is an increasing function. Therefore \( \text{Max}F(\delta) = F(1) \).

Consequently,
\[ |a_2a_4 - a_3^2| \leq \frac{1}{288} G(p) , \quad (19) \]

where \( G(p) = F(1) \). So,
\[ G(p) = A(\alpha) p^4 - 4B(\alpha) p^2 + 32 (3 - \alpha)^2 , \]

where
\[ A(\alpha) = 2 (2\alpha^4 - 12\alpha^3 + 15\alpha^2 - 18\alpha + 30) , \]

and
\[ B(\alpha) = (-2\alpha^3 + 13\alpha^2 - 66\alpha + 96) , \]

Now
\[ G'(p) = 4A(\alpha) p^3 - 8B(\alpha) p \]

and
\[ G''(p) = 12A(\alpha) p^2 - 8B(\alpha) \]

then \( G'(p) = 0 \) gives
\[ p [4A(\alpha) p^2 - 8B(\alpha)] = 0. \]

\( G''(p) \) is negative at
\[ p = \sqrt{\frac{96 - 66\alpha + 13\alpha^2 - 2\alpha^3}{30 - 18\alpha + 15\alpha^2 - 12\alpha^3 + 2\alpha^4}} = p'. \]

So \( \text{Max}G(p) = G(p') \). Hence from (19), we obtain (3.1).

The result is sharp for \( p_1 = p', p_2 = p_1^2 - 2 \) and \( p_3 = p_1 (p_1^2 - 3) \).

For \( \alpha = 1 \) and \( \alpha = 0 \) respectively, we obtain the following results:

**Corollary 3.2.** If \( g(z) \in RT \), then
\[ |a_2a_4 - a_3^2| \leq \frac{4}{9}. \]

**Remark 3.3.** For the choice of \( \alpha = 1 \), the result coincides with those of A. Janteng, S.A. Halim and M. Darus ([12], [13]).
Corollary 3.3. If $g(z) \in CC$, then

$$|a_2a_4 - a_3^2| \leq 1.$$ 

Theorem 3.4. If $f \in C'_*(\alpha)$, then

$$|a_2a_4 - a_3^2| \leq \frac{(7 - \alpha)^2}{324}. \quad (20)$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1(p_1^2 - 3)$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.5. If $h(z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}.$$ 

Corollary 3.6. If $h(z) \in C'$, then

$$|a_2a_4 - a_3^2| \leq \frac{49}{324}.$$ 

4. Functions with Respect to Symmetric Points

Theorem 4.1. If $f \in C_s^{*(\alpha)}$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}. \quad (21)$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.2. If $f(z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$ 

Corollary 4.3. If $f(z) \in S_s^{*}$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$
Theorem 4.4. If $C^\alpha_s$, then

$$|a_2a_4 - a_3^2| \leq \frac{(3 - \alpha)^2}{9}. \quad (22)$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.5. If $g(z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$ 

Corollary 4.6. If $g(z) \in S^*_s$, then

$$|a_2a_4 - a_3^2| \leq 1.$$ 

Theorem 4.7. If $C^\alpha_{1(s)}$, then

$$|a_2a_4 - a_3^2| \leq \frac{(7 - \alpha)^2}{81}. \quad (23)$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.8. If $h(z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$ 

Corollary 4.9. If $h(z) \in C'_s$, then

$$|a_2a_4 - a_3^2| \leq \frac{49}{81}.$$ 

References


