Abstract: In this paper we present a new method for constructing Halley’s method. The idea is based on the Newton method, i.e. we change the equation such that applying the Newton method to the new one has at least cubic convergence order.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as the Newton method, are usually used. Throughout this paper we consider an iterative method to find a simple root \( \alpha \) of a nonlinear equation \( f(x) = 0 \), where \( f : I \subseteq \)
$R \to R$, for an open interval $I$, $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. The Newton method for the calculation of $\alpha$ is probably the most widely used iterative scheme, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

(1)

It is well known that this method is quadratically convergent [6]. Some modifications of the Newton method to achieve higher order of convergence and better efficiency have been suggested and analyzed in the literature, see e.g. the books by Ostrowsky [4], Turab [6], and Neta [2]. Most of the methods improve the order of convergence and computational efficiency of Newton method with an additional evaluation of the function or it’s derivatives. To measure a balance between the number of function evaluations and the order of convergence of the method, we have the following definitions:

- **Informational efficiency**

  We define informational efficiency $E$ by $E = \frac{p}{d}$, where $p$ is the order of convergence and $d$ is the number of function (or derivative) evaluations per step [6].

- **Efficiency index**

  The efficiency index $E^* = p^{\frac{1}{d}}$, has been introduced by Ostrowsky [4].

To be more precise, we have another measure by Traub [6], called computational efficiency that includes the computational price for each derivative of the function. One can see [6] for more details.

### 2. A New Idea for Constructing the Halley’s Method

A well known method for solving nonlinear equations is the Halley’s method [3]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2},$$

(2)

which has cubic convergence order. As we know the order of convergence in Newton’s method for simple roots ($f(\alpha) = 0$, $f'(\alpha) \neq 0$) is at least 2. If additional condition $f''(\alpha) = 0$ is satisfied then the order of convergence for Newton’s method is at least 3, [5]. Our idea is to define an artificial function $g(x)$ and solve $h(x) = g(x)f(x) = 0$. Now we must define $g(x)$ such that $h(\alpha) = 0$, $h'(\alpha) \neq 0$, $h''(\alpha) = 0$, (which $\alpha$ is the root of $f(x)$) so the order of convergence for the Newton’s method in $h(x) = 0$ is at least 3. $h(\alpha) = 0$ is satisfied because,
\( f(\alpha) = 0. \) \( h'(\alpha) - f'(\alpha)g(\alpha) + g'(\alpha)f(\alpha) = f'(\alpha)g(\alpha), \) \( \alpha \) is the simple root of \( f, \) so The first condition on \( g \) is that \( g(\alpha) \neq 0, \) finally we need to have \( f''(\alpha) = 0 \) or \( f''(\alpha)g(\alpha) + 2f'(\alpha)g'(\alpha) + g''(\alpha)f(\alpha) = 0, \) or

\[ f''(\alpha)g(\alpha) = -2f'(\alpha)g'(\alpha). \quad (3) \]

Considering the above conditions we apply Newton’s method to \( h(x) = 0, \) the iterations are as follow:

\[ x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}, \quad (4) \]

\[ x_{n+1} = x_n - \frac{f(x_n)g(x_n)}{f'(x_n)g(x_n) + f(x_n)g'(x_n)}. \quad (5) \]

From (3) we have:

\[ g'(\alpha) = \frac{f''(\alpha)g(\alpha)}{-2f'(\alpha)}. \quad (6) \]

So:

\[ g'(x_n) \approx \frac{f''(x_n)g(x_n)}{-2f'(x_n)}. \quad (7) \]

Putting (7) in (5) we have

\[ x_{n+1} = x_n - \frac{f(x_n)g(x_n)}{f'(x_n)g(x_n) + f(x_n)\frac{f''(x_n)g(x_n)}{-2f'(x_n)}}, \quad (8) \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}. \quad (9) \]

Which is the well known Halley’s method.

3. Conclusions

In this paper we use a new idea for developing Newton’s method, which led to Halley’s method. One can use the idea for constructing new efficient methods with higher order of convergence.

References


