International Journal of Applied Mathematics

Volume 28 No. 5 2015, 515-525

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i5.5

SOLVING THE IVANCEVIC OPTION PRICING MODEL USING THE ELSAKI-ADOMIAN DECOMPOSITION METHOD

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Abstract: In this paper, we propose a new computational algorithm called Elsaki-Adomian Decomposition Method (EADM) to solve a nonlinear Black-Scholes model recently established in [9], [10]. The proposed method is a combination of the Elsaki transform (ET) and the Adomian Decomposition Method (ADM). This new computational algorithm is applied directly without using any transformation, linearization, discretization or taking any additional restrictive assumption. Exact solutions of an illustrative example are successfully found by using the proposed method in order to show its reliability, efficiency, and accuracy.

AMS Subject Classification: 91G80, 35L05, 44A05
Key Words: Ivancevic option pricing, Adomian decomposition method (ADM), Elsaki transform, Adomian polynomials

1. Introduction

Recently, Ivancevic [9, 10], based on the modern adaptive markets hypothesis

Received: June 15, 2015

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[11, 12] the wave market theory [8, 13] and the quantum neural computation approach, proposed a novel nonlinear option pricing model, which we will call the Ivancevic option pricing model, specified as

$$iV_t + \frac{1}{2}\sigma^2 V_{SS} + \beta|V|^2 V = 0, \tag{1.1}$$

in order to satisfy efficient and behavioral markets, and their underlying essential nonlinear complexity, where $V = V(S,t)$ denotes the option-price wave function, σ is the dispersion frequency coefficient the volatility, (which can be either a constant or stochastic process itself) and β is the Landau coefficient which represents the adaptive market potential.

We use a plane wave as an initial solution of the (1.1), i.e. we solve the problem of initial value

$$\begin{cases} iV_t + \frac{1}{2}\sigma^2 V_{SS} + \beta|V|^2 V = 0, \\ V(S,0) = f(S), \end{cases} \tag{1.2}$$

where σ and β are constant.

In this paper we present some basic definitions of the Elzaki transform (ET) and the Adomian decomposition method (ADM) and present a reliable combination of the Adomian decomposition method and the Elzaki transform, which we will name as Elsaki-Adomian Decomposition Method (EADM), to solve the Cauchy problem given by (1.2). The problem (1.2) has also been studied in [15]. Finally, we will discuss a couple of examples to illustrate the accuracy and usefulness of the method we propose.

2. The Elsaki-Adomian Decomposition Method

In this section we present some basic definitions of the Elzaki transform and Adomian Decomposition Method (ADM) and present a reliable combination of the ADM and the Elzaki transform to obtain the solution of nonlinear partial differential equations; specifically, we solve a nonlinear Black-Scholes model recently established in [9] and [10].

2.1. Basic Definitions and Properties of the Elsaki Transform

The definition of the Elsaki transform and their properties have been recently established in [7].

Definition 1. The Elsaki transform of the functions $f : [0, \infty) \to \mathbb{R}$ belonging to a class $\mathcal{A}$ is defined as

$$E\{f(t)\} = F(u) = u \int_0^\infty f(t) e^{-\frac{t}{u}} dt, \quad u \in (k_1, k_2).$$

In the above definition,

$$\mathcal{A} = \{f : \exists M, k_1, k_2 > 0 \text{ s. t., for each } t \in [0, \infty),\ |f(t)| < M e^{\frac{t}{k_j}}\}.$$

It is easy to see that the Elsaki transform is a linear operator and we have the following proposition.

Proposition 1. *Let f be a real-valued function defined on $\mathbb{R} \times [0, \infty)$ at least twice differentiable with respect to the second variable, $f \in \mathcal{A}$, then*

(i) $E\left\{\dfrac{\partial f(x,t)}{\partial t}\right\} = \dfrac{1}{u} T(x,u) - u f(x,0)$,

(ii) $E\left\{\dfrac{\partial^2 f(x,t)}{\partial t^2}\right\} = \dfrac{1}{u^2} T(x,u) - f(x,0) - u \dfrac{\partial f(x,0)}{\partial t}$.

Proof. To obtain transforms of the partial derivatives we use integration by parts as follows:

$$E\left\{\frac{\partial f(x,t)}{\partial t}\right\} = u \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{u}} dt = \lim_{a \to \infty} \int_0^a u e^{-\frac{t}{u}} \frac{\partial f(x,t)}{\partial t} dt$$

$$= \lim_{a \to \infty} \left(\left[u e^{-\frac{t}{u}} f(x,t) \right]_0^a - \int_0^a e^{-\frac{t}{u}} f(x,t) dt \right) = \frac{1}{u} T(x,u) - u f(x,0);$$

thus we have shown (i). Now to prove (ii), we assume that f is at last twice differentiable with respect to t and that $f \in \mathcal{A}$. Let $h(x,t) = \frac{\partial f(x,t)}{\partial t}$, then we obtain

$$E\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = E\left\{\frac{\partial h(x,t)}{\partial t}\right\} = \frac{1}{u} E\left\{h(x,t)\right\} - u h(x,0),$$

and therefore, using (i):

$$E\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \frac{1}{u^2} T(x,u) - f(x,0) - u \frac{\partial f(x,0)}{\partial t}.$$

□

Following Definition 1 directly we can obtain the following table:

$f(t)$	$E\{f(t)\} = F(u)$
1	u^2
t	u^3
t^n	$n!u^{n+2}$
e^{at}	$\frac{u^2}{1-au}$
$\sin(at)$	$\frac{au^3}{1+a^2u^2}$
$\cos(at)$	$\frac{u^2}{1+a^2u^2}$

Table 1: Elsaki transform of some functions.

2.2. The Adomian Decomposition Method

The Adomian Decomposition Method (ADM) is a technique for solving ordinary and partial nonlinear differential equations [3]. Using this method, it is possible to express analytic solutions in terms of a rapidly converging series [4]. In a nutshell, the method identifies and separates the linear and nonlinear parts of a differential equation. By inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the the rest of the equation affected by this inverse operator. Here, we propose to express this solution by means of a decomposition series with terms that will be well determined by recursion and that gives rise to the solution components.

Given a partial (or ordinary) differential equation

$$Fu(x,t) = g(x,t) \tag{2.1}$$

with the initial condition

$$u(x,0) = f(x), \tag{2.2}$$

where F is a differential operator that could be, in general, nonlinear and therefore includes linear and nonlinear terms.

Equation (2.1) can be written as

$$L_t u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{2.3}$$

where $L_t = \frac{\partial}{\partial t}$, R is the linear remainder operator that could include partial derivatives with respect to x, N is a nonlinear operator which is presumed to be analytic and g is a non-homogeneous term that is independent of the solution u.

Solving for $L_t u(x,t)$, we have

$$L_t u(x,t) = g(x,t) - Ru(x,t) - Nu(x,t). \tag{2.4}$$

As L is presumed to be invertible, we can apply $L_t^{-1}(\cdot) = \int_0^t (\cdot)dt$ to both sides of equation (2.4) obtaining

$$L_t^{-1} L_t u(x,t) = L_t^{-1} g(x,t) - L_t^{-1} Ru(x,t) - L_t^{-1} Nu(x,t). \tag{2.5}$$

An equivalent expression to (2.5) is

$$u(x,t) = f(x) + L_t^{-1} g(x,t) - L_t^{-1} Ru(x,t) - L_t^{-1} Nu(x,t), \tag{2.6}$$

where $f(x)$ is the constant of integration with respect to t that satisfies $L_t f = 0$. In equations where the initial value $t = t_0$, we can conveniently define L^{-1}.

The ADM proposes a decomposition series solution $u(x,t)$ given as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.7}$$

The nonlinear term $Nu(x,t)$ is given as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{2.8}$$

where $\{A_n\}_{n=0}^{\infty}$ is the Adomian polynomials sequence [14] given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\alpha^n} [N(\sum_{k=0}^{n} \alpha^k u_k)]|_{\alpha=0}. \tag{2.9}$$

Substituting (2.7), (2.8) y (2.9) into equation (2.6), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1} g(x,t) - L_t^{-1} R \sum_{n=0}^{\infty} u_n(x,t) - L_t^{-1} \sum_{n=0}^{\infty} A_n, \tag{2.10}$$

from which we can establish the following recurrence relation, for every $n = 0, 1, 2, \ldots$, we have:

$$\begin{cases} u_0(x,t) = f(x) + L_t^{-1} g(x,t), \\ u_{n+1}(x,t) = L_t^{-1} Ru_n(x,t) - L_t^{-1} A_n(u_0, u_1, \ldots, u_n), \end{cases} \tag{2.11}$$

Using the recurrence formulas (2.11), we can obtain an approximate solution of (2.1), (2.2) as

$$u_k(x,t) = \sum_{n=0}^{k} u_n(x,t), \quad \text{where} \lim_{k\to\infty} u_k(x,t) = u(x,t). \tag{2.12}$$

In many cases, the convergence of this series is very fast and only a few terms are needed in order to have an idea of how the solutions behave. Convergence conditions of this series are examined in [5], [6], [1] and [2] mainly.

3. The Ivancevic Option Pricing Model Using the EADM

Let us consider the initial value problem given in the Ivancevic option pricing equation (1.2) with a plane wave as an initial solution, i.e. we solve the problem

$$\begin{cases} iV_t + \frac{1}{2}\sigma^2 V_{SS} + \beta|V|^2V = 0, \\ V(S,0) = e^{iS}, \end{cases} \tag{3.1}$$

here we are considering $\sigma \neq 0$ and $\beta \neq 0$ constant and set $\frac{2\beta}{\sigma^2} = \alpha$.

By applying the Elsaki transform on both sides of the equation (3.1) and using the linearity of the Elsaki transform,

$$\frac{1}{u}F(S,u) - uV(S,0) - iE\{V_{SS}\} - iE\{\alpha|V|^2V\} = 0. \tag{3.2}$$

Substituting the given initial condition of equation (3.1) into (3.2), we obtain

$$\frac{1}{u}F(S,u) - ue^{iS} - iE\{V_{SS}\} - iE\{\alpha|V|^2V\} = 0, \tag{3.3}$$

where, solving for $F(S,u)$ we obtain

$$F(S,u) = u^2e^{iS} + iuE\{V_{SS} + \alpha|V|^2V\}. \tag{3.4}$$

Now, taking the inverse Elsaki transform of (3.4), we obtain

$$V(S,t) = e^{iS} + iE^{-1}[uE\{V_{SS} + \alpha|V|^2V\}], \tag{3.5}$$

where $V(S,t)$ is the unknown function of the nonlinear term $\alpha|V|^2V$.

Now, using (2.7), we decompose the unknown function $V(S,t)$ as a sum of components defined by the series

$$V(S,t) = \sum_{n=0}^{\infty} V_n(S,t), \tag{3.6}$$

and the nonlinear term can be represented by an infinite series

$$|V|^2V = V^2\bar{V} = \sum_{n=0}^{\infty} A_n(V_0, V_1, \ldots, V_n) \tag{3.7}$$

where the components A_n are the Adomian polynomials which can be calculated by formula (2.9) as follows:
$A_0 = V_0^2\bar{V}_0$,
$A_1 = 2V_0V_1\bar{V}_0 + V_0^2\bar{V}_1$,
$A_2 = 2V_0V_2\bar{V}_0 + V_1^2\bar{V}_0 + 2V_0V_1\bar{V}_1 + V_0^2\bar{V}_2$,
$A_3 = 2V_0V_3\bar{V}_0 + 2V_1V_2\bar{V}_0 + 2V_0V_2\bar{V}_1 + V_1^2\bar{V}_1 + 2V_0V_1\bar{V}_2 + V_0^2\bar{V}_3$,
$\vdots \qquad \vdots$
By substituting (3.6) and (3.7) into (3.5),

$$\sum_{n=0}^{\infty} V_n(x,t) = e^{iS} + iE^{-1}\left[uE\left\{\sum_{n=0}^{\infty} V_{n,SS} + \alpha\sum_{n=0}^{\infty} A_n\right\}\right]. \tag{3.8}$$

By comparing the both sides of the Eq. (3.8) we have the recursive formula

$$\begin{cases} V_0(S,t) = e^{iS}, \\ V_{k+1}(S,t) = iE^{-1}\Big[uE\left\{V_{k,SS} + \alpha A_k\right\}\Big], \quad k \geq 0. \end{cases} \tag{3.9}$$

Hence, the exact or approximate solutions of the problem (3.1) is given by

$$V(S,t) = \sum_{n=0}^{\infty} V_n(S,t).$$

3.1. Illustrative Examples

In this subsection, some initial value problems are presented to show the reliability, efficiency and accuracy of the proposed method.

Example 1. Consider the following nonlinear Ivancevic option pricing model:

$$\begin{cases} iV_t + V_{SS} + 6|V|^2V = 0, \\ V(S,0) = e^{2iS}, \end{cases} \tag{3.10}$$

where $\sigma^2 = 1$, $\beta = 3$ and therefore $\alpha = 6$.

Using the recursive relation (3.9), we have

$$V_0(S,t) = e^{2iS}, \tag{3.11}$$

$$V_{k+1}(S,t) = iE^{-1}\Big[uE\left\{V_{k,SS} + 6A_k\right\}\Big], \quad k \geq 0, \tag{3.12}$$

where A_n is an Adomian polynomial which represent the nonlinear term $|V|^2V$. Now using the recursive relation of equation (3.12), we can easily compute the other terms of the $V(S,t)$ as follows:

$$\begin{aligned} V_1(S,t) &= iE^{-1}\Big[uE\left\{V_{0,SS} + 6A_0\right\}\Big] = iE^{-1}\Big[uE\left\{V_{0,SS} + 6V_0^2\bar{V}_0\right\}\Big] \\ &= iE^{-1}\big[uE\left\{2e^{2iS}\right\}\big] = 2ie^{2iS}E^{-1}[uE\left\{1\right\}] = 2ie^{2iS}E^{-1}[u^3] \\ &= 2ite^{2iS}, \end{aligned}$$

in a similar way,

$$V_2(S,t) = iE^{-1}\Big[uE\left\{V_{1,SS} + 6A_1\right\}\Big] = -2t^2e^{2iS},$$

$$V_3(S,t) = iE^{-1}\Big[uE\left\{V_{2,SS} + 6A_2\right\}\Big] = -\frac{8i}{3!}t^3e^{2iS},$$

$$V_4(S,t) = iE^{-1}\Big[uE\left\{V_{3,SS} + 6A_3\right\}\Big] = \frac{16}{4!}e^{2iS}t^4,$$

and so on.

Therefore, the series solution of the function $V(S,t)$ is given by:

$$\begin{aligned} V(S,t) &= \sum_{n=0}^{\infty} V_n(S,t) \\ &= V_0(S,t) + V_1(S,t) + V_2(S,t) + V_3(S,t) + V_4(S,t) + \cdots \\ &= e^{2iS}\Big(1 + 2it + \frac{(2it)^2}{2!} + \frac{(2it)^3}{3!} + \frac{(2it)^4}{4!} + \cdots\Big) = e^{2i(S+t)}. \end{aligned}$$

It is straightforward to verify that $V(S,t) = e^{2i(S+t)}$ satisfies the problem (3.10).

Example 2. We consider the special case of a S-independent nonlinear Ivancevic option pricing model:

$$\begin{cases} iV_t + V_{SS} + 6|V|^2V = 0, \\ V(S,0) = 1. \end{cases} \tag{3.13}$$

Using the recursive relation (3.9), we have

$$V_0(S,t) = 1, \tag{3.14}$$

$$V_{k+1}(S,t) = iE^{-1}\Big[uE\left\{V_{k,SS} + 6A_k\right\}\Big], \quad k \geq 0, \tag{3.15}$$

where A_n is an Adomian polynomial which represent the nonlinear term $|V|^2V$. Now we will use the recursive relation of equation (3.15), we can easily compute the other terms of the $V(S,t)$ as follows:

$$\begin{aligned} V_1(S,t) &= iE^{-1}\Big[uE\left\{V_{0,SS} + 6A_0\right\}\Big] = iE^{-1}\Big[uE\left\{6V_0^2\bar{V}_0\right\}\Big] \\ &= iE^{-1}\big[uE\left\{6\right\}\big] = 6iE^{-1}[uE\left\{1\right\}] = 6iE^{-1}[u^3] = 6it, \end{aligned}$$

in a similar way,

$$V_2(S,t) = iE^{-1}\Big[uE\left\{V_{1,SS} + 6A_1\right\}\Big] = -\frac{36}{2!}t^2,$$

$$V_3(S,t) = iE^{-1}\Big[uE\left\{V_{2,SS} + 6A_2\right\}\Big] = -\frac{216i}{3!}t^3,$$

$$V_4(S,t) = iE^{-1}\Big[uE\left\{V_{3,SS} + 6A_3\right\}\Big] = \frac{1296}{4!}t^4,$$

and so on.

Therefore, the series solution of the function $V(S,t)$ is given by:

$$\begin{aligned} V(S,t) &= \sum_{n=0}^{\infty} V_n(S,t) \\ &= V_0(S,t) + V_1(S,t) + V_2(S,t) + V_3(S,t) + V_4(S,t) + \cdots \\ &= 1 + 6it + \frac{(6it)^2}{2!} + \frac{(6it)^3}{3!} + \frac{(6it)^4}{4!} + \cdots = e^{6it}. \end{aligned}$$

It is very easy to verify that $V(S,t) = e^{6it}$ satisfies the problem (3.13).

4. Conclusion

The main goal of this paper was to present a reliable combination of Adomain decomposition and the Elzaki transform. The proposed method is powerful and efficient. An important conclusion that we can make is that the Elsaki-Adomain decomposition methods for solving the nonlinear partial differential equations (1.2) provide the components of exact solution.

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