ON THE GEOMETRY OF CLOSED TIMELIKE RULED SURFACES IN DUAL LORENTZIAN SPACE

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Abstract: In this paper, a dual timelike curve $c(t)$ which is a Lorentzian spherical indicatrix of a timelike closed ruled surface $(\vec{V}_1^*(t))$ with a real parameter $t$, two frames $D^*(\vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*)$ related to the timelike closed ruled surface $(\vec{V}_1^*(t))$ and $D(\vec{V}_1, \vec{V}_2, \vec{V}_3)$ which moves respect to $D^*$ and related to a timelike closed ruled surface drawn by a timelike vector $\vec{V}_1(t)$, and a timelike vector $\vec{V}$ which is fixed in the frame $D$ are considered; and dual integral invariants of the timelike closed ruled surfaces which correspond to the dual timelike closed curves drawn by the vectors $\vec{V}$, $\vec{V}_1$ and $\vec{V}_1^*$ are studied; and it is found same relations among the dual integral invariants of the timelike closed ruled surfaces which correspond to the dual timelike closed curves drawn by the timelike vectors $\vec{V}$, $\vec{V}_1$ and $\vec{V}_1^*$. In addition, these results are carried to the Lorentzian line space $\mathbb{R}^3_1$ and give some theorems by means of Study’s mapping.

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1. Introduction

The geometry of ruled surface is very important in the study of kinematics or spatial mechanism, surface design, manufacturing technology, robotic research in $\mathbb{R}^3$, [4,5,6,7,8,22]. It is well known, from [10] that a $V_1$− closed ruled surface generated by $V_1$− oriented line of rigid body have two real integral invariants; the real pitch $l_{V_1}$ and the real angle of $\lambda_{V_1}$.

In recent years, by using the real integral invariants of closed ruled surfaces, many investigations are done, [3,9,23]. In addition, by using Lorentzian metric, ruled surfaces in Lorentz space have been studied [1,11,12,15,16,20].

In this paper, a dual closed spherical motion given by [24] is considered in the dual Lorentzian 3− space $\mathbb{D}^3_1$. During this motion, same relations among the dual integral invariants of the timelike ruled surfaces drawn by the timelike vectors $\vec{V}$, $\vec{V}_1$ and $\vec{V}_1^*$ are given. In addition, separating real and dual parts of these dual integral invariants, some results and some theorems related to Holditch Theorem are given in the Lorentzian 3-space $\mathbb{R}^3_1$ by Study’s mapping.

2. Basic Concepts

We recall some fundamental concepts of the subject. If $a$ and $a^*$ are real numbers and $\epsilon^2 = 0$, the combination $A = a + \epsilon a^*$ is called a dual number, where $\epsilon$ is dual unit. The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$.

Let us consider Lorentzian 3− space $\mathbb{R}^3_1 = [\mathbb{R}^3, (-, +, +)]$ and the Lorentzian inner product of $\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3) \in \mathbb{R}^3_1$ be $\langle \vec{A}, \vec{B} \rangle = -a_1b_1 + a_2b_2 + a_3b_3$.

The vector $\vec{A} = (a_1, a_2, a_3) \in \mathbb{R}^3_1$ is said to be timelike if $\langle \vec{A}, \vec{A} \rangle < 0$, spacelike if $\langle \vec{A}, \vec{A} \rangle > 0$ or $\vec{A} = 0$ and lightlike (null) if $\langle \vec{A}, \vec{A} \rangle = 0$ and $\vec{A} \neq \vec{0}$.

Consider moving unit dual Lorentzian sphere $K$ generated by a dual orthonormal system $(0; \vec{V}_1, \vec{V}_2, \vec{V}_3)$ and fixed unit dual Lorentzian sphere generated by a dual orthonormal system $(0; \vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*)$.

**Definition 1.** Consider a non-singular linear transformation between two dual orthonormal coordinate systems linked to the unit dual Lorentzian spheres $K$ (moving) and $K_1$ (fixed), respectively. In the case that the corresponding matrix is an orthogonal matrix whose elements are differentiable dual func-
tions of a real parameter $t$, the transformation is called a one-parameter dual Lorentzian spherical motion and denoted by $K/K_1$.

**Theorem 2.** ([19]) There exists one-to-one correspondence between directed timelike (resp. spacelike) lines of $\mathbb{R}^3_1$ and an ordered pair of vectors $(\vec{a}, \vec{a}^*)$ such that $<\vec{a}, \vec{a}> = -1$ (resp. $<\vec{a}, \vec{a}> = 1$) and $<\vec{a}, \vec{a}^*> = 0$.

**Definition 3.** A timelike closed ruled surface $(X)$ is one-to-one correspondence with a dual closed curve on the Lorentzian sphere, [18].

Let \( \{\vec{V}_1, \vec{V}_2, \vec{V}_3\} \) and \( \{\vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*\} \) be two right-handed sets of orthonormal unit dual vectors that are rigidly linked to the Lorentzian dual spheres $K$ and $K_1$, and denoted by

\[
\vec{V} = \begin{pmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \end{pmatrix}, \quad \vec{V}^* = \begin{pmatrix} \vec{V}_1^* \\ \vec{V}_2^* \\ \vec{V}_3^* \end{pmatrix},
\]

respectively. Between frames $\vec{V}$ and $\vec{V}^*$, the relation

\[
\Omega = \begin{pmatrix} 0 & W^2_1 & W^3_1 \\ W^2_1 & 0 & -W^3_2 \\ W^3_1 & W^3_2 & 0 \end{pmatrix}
\]

is given by [13]. We assume that $\vec{V}_3, \vec{V}_2, \vec{V}_3^*$ and $\vec{V}_2^*$ are spacelike dual vectors, $\vec{V}_1$ and $\vec{V}_1^*$ are timelike dual vectors in $D^3_1$.

**Definition 4.** Let $\vec{\psi}$ be a dual pfaf vector during the dual motion $K/K_1$. The dual vector $\vec{D}$ is defined by

\[
\vec{D} = \oint \vec{\psi} = \oint (W^3_2 \vec{V}_1 + W^3_1 \vec{V}_2 - W^2_1 \vec{V}_3) = \vec{d} + \epsilon \vec{d}^*
\]

is called dual Steiner vector of the dual motion $K/K_1$, [13].

**Definition 5.** The dual angle of pitch of a timelike closed ruled surface $(X(t))$ is given by [17] as

\[
\Lambda_x = -<\vec{D}, \vec{X}>
\]
3. Dual Angles of Pitch of Closed Ruled Surfaces Drawn on the Fixed Dual Sphere $K_1$ by Vectors $\vec{V}, \vec{V}_1,$ and $\vec{V}_1^*$

Let us choose a fixed dual point $\vec{V}_1^*$ which is on the unit dual moving Lorentzian sphere $K$. During the one-parameter dual Lorentzian spherical closed motion $K/K_1$, for every $t$, there is a constant dual angle $\phi$ between the vector $\vec{V}_1^*$ and unit dual vector $\vec{V}_1$ of the moving Lorentzian sphere $K$, i.e., let us choose

$$< \vec{V}_1, \vec{V}_1^* > = \cosh \phi.$$ (5)

During the dual closed Lorentzian spherical motion $K/K_1$, $\vec{V}_1$ and $\vec{V}_1^*$ draw two closed spherical curves on the unit dual fixed Lorentzian sphere $K_1$. These curves correspond to two closed ruled surfaces in the fixed line space $H_1$ by Study’s mapping. Let us denote the closed ruled surfaces drawn by the vectors $\vec{V}_1$ and $\vec{V}_1^*$ by $(\vec{V}_1)$ and $(\vec{V}_1^*)$, respectively.

Consider $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ and $\{\vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*\}$ two sets of orthonormal unit dual vectors that are rigidly linked to the Lorentzian spheres $K$ and $K_1$, and denoted by

$$D(\vec{V}_1, \vec{V}_2, \vec{V}_3), \quad D^*(\vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*)$$ (6)

respectively, where frames $D$ and $D^*$ are taken form by the vectors $\vec{V}_1$ and $\vec{V}_1^*$, respectively.

These two frames are rigidly linked to $D$ and $D^*$ by

$$\vec{V}_3 = \frac{\vec{v}_1 \times \vec{v}_1^*}{\sinh \phi},$$
$$\vec{V}_2 = \vec{V}_3 \times \vec{V}_1,$$
$$\vec{V}_3^* = -\vec{V}_3^*,$$
$$\vec{V}_2^* = \vec{V}_3^* \times \vec{V}_1^*.$$ (7)

Thus, for the vectors $\vec{V}_1$ and $\vec{V}_1^*$ we have

$$\vec{V}_1^* = \cosh \phi \vec{V}_1 + \sinh \phi \vec{V}_2$$
$$\vec{V}_2^* = \sinh \phi \vec{V}_1 + \cosh \phi \vec{V}_2,$$ (8)

or

$$\vec{V}_1 = \cosh \phi \vec{V}_1^* + \sinh \phi \vec{V}_2^*$$
$$\vec{V}_2 = \sinh \phi \vec{V}_1^* + \cosh \phi \vec{V}_2^*,$$
$$\vec{V}_3 = -\vec{V}_3^*, \quad (\phi = \varphi + \epsilon \varphi^*)$$ (9)
Let us denote the dual hyperbolic angle of the vectors $\vec{V}_1$ and $\vec{V}$ by $\Delta$ and the angle of the vectors $\vec{V}_1$ and $\vec{V}_1^*$ by $\phi$.

Let us consider an unit dual timelike vector $\vec{V}$ which is fixed in the orthonormal frame $D$ and expressed as follows:

$$\vec{V} = \cosh \Delta \vec{V}_1 + \sinh \Delta \cos \Theta \vec{V}_2 + \sinh \Delta \sin \Theta \vec{V}_3,$$

(10)

$$(\Delta = \alpha + \epsilon \alpha^*, \Theta = \theta + \epsilon \theta^*),$$

where $\vec{V}_1$ is timelike.

In addition, unit dual timelike vector $\vec{V}$ with respect to dual orthonormal frame $D^*$ by equations (9) and (10) also is written as follows:

$$\vec{V} = (\cosh \Delta \cosh \phi + \sinh \Delta \sinh \phi \cos \Theta)\vec{V}_1^*$$

$$+ (\cosh \Delta \sinh \phi + \sinh \Delta \cosh \phi \cos \Theta)\vec{V}_2^* - \sinh \Delta \sin \Theta \vec{V}_3^*.$$

(11)

By means of equation (10)

$$<\vec{V}_1, \vec{V}> = \cosh \Delta.$$ (12)

In the same way

$$<\vec{V}_1^*, \vec{V}> = \cosh W.$$ (13)

Equation (13) by means of equation (11) is written as follows:

$$\cosh W = \cosh \Delta \cosh \phi + \sinh \Delta \sinh \phi \cos \Theta.$$ (15)

For $\Theta = 0$, from equation (13)

$$\cosh W = \cosh \Delta \cosh \phi + \sinh \Delta \sinh \phi = \cosh(\Delta + \phi),$$ (14)

or $W = \Delta + \phi$ is obtained.

In addition, constant dual hyperbolic angles $\Delta_i$ and $W_i$ which are among the unit dual vectors $(\vec{V}_i, \vec{V})$ and $(\vec{V}_i^*, \vec{V})$ are given as follows:

$$\cosh \Delta_i = <\vec{V}_i, \vec{V}>$$

and

$$\cosh W_i = <\vec{V}_i^*, \vec{V}>, \quad (i = 1, 2, 3),$$ (15)

respectively.

From equations (10) and (11), the relations among constant angles $\Delta_i$ and $W_i$ as depend on the angles $\Delta, \phi$ and $\Theta$ are obtained.
Thus, during the dual Lorentzian motion $K/K_1$ we may consider the timelike closed ruled surface drawn by the timelike vector $\vec{V}$ which is fixed in the orthonormal frames $D$ and $D^*$. This timelike closed ruled surface is expressed as $(\vec{V})$.

Dual angle of pitch of the timelike closed ruled surface $(\vec{V})$ by means of equations (3) and (4) is written as follows:

$$\Lambda_V = - \langle \vec{D}, \vec{V} \rangle.$$ (16)

Thus, from equations (10) and (16)

$$\Lambda_V = \cosh \Delta \oint W_2^3 - \sinh \Delta \cos \Theta \oint W_1^3 + \sinh \Delta \sin \Theta \oint W_1^2$$ (17)

is obtained.

The integral $\oint W_2^3$ is the dual angle of pitch of the timelike closed ruled surface $(\vec{V}_1)$. In this case

$$\Lambda_{V_1} = \oint W_2^3 = - \langle \vec{D}, \vec{V}_1 \rangle,$$ (18)

may be written.

In addition, because of this also stated the unit dual timelike vector $\vec{V}$ according to the frame $D^*$, in the same way

$$\Lambda_V = - \langle \vec{D}^*, \vec{V}^* \rangle,$$ (19)

may be written, where $\vec{D}^*$ is given as follows:

$$\vec{D}^* = \oint [(W_2^3)^* \vec{V}_1 + (W_1^3)^* \vec{V}_2^* - (W_1^2)^* \vec{V}_3^*].$$ (20)

The integral $\oint (W_3^3)$ may be consider as a function of the dual angle of pitch $\Lambda_{V_1}$ and $\Lambda_{V_1^*}$ of the timelike closed ruled surfaces $(\vec{V}_1)$ and $(\vec{V}_1^*)$, respectively. For this case, we consider the dual angle of pitch of the closed ruled surface $(\vec{V}_1^*)$. Thus, from the equations (18) and (16), $\Lambda_{V_1^*}$ is written as follows:

$$\Lambda_{V_1^*} = - \langle \vec{D}^*, \cosh \phi \vec{V}_1 + \sinh \phi \vec{V}_2 \rangle,$$

or

$$\Lambda_{V_1^*} = \cosh \phi \oint W_2^3 - \sinh \phi \oint W_1^3.$$ (21)
Thus, equation (21) is written as follows:

$$\Lambda_{V_1^*} = \Lambda_{V_1} \cosh \phi - \sinh \phi \oint W_1^3.$$  (22)

From equation (22), for the integral $-\oint W_1^3$ we have

$$-\oint W_1^3 = \frac{1}{\sinh \phi} (\Lambda_{V_1^*} - \Lambda_{V_1} \cosh \phi).$$  (23)

For meaning of integral $\oint W_1^2$ we use equalities in the following:

$$\oint W_1^2 = \oint [d, W_1^2] = \oint [W_3, W_1^2] = a_{V_3}$$  (24)

is obtained, where $\partial G_3$ is the dual Lorentzian spherical area bounded on the fixed dual Lorentzian sphere by the dual Lorentzian spherical indicatrix of the spacelike vector $\vec{V}_3$ [19]. Thus, from equations (17), (18), (23) and (24)

$$\Lambda_V = \Lambda_{V_1} \cosh \Delta + \frac{\sinh \Delta \cos \Theta}{\sinh \phi} (\Lambda_{V_1^*} - \Lambda_{V_1} \cosh \phi) + \sinh \Delta \sin \Theta a_{V_3},$$

or

$$\Lambda_V = \frac{1}{\sinh \phi} (\cosh \Delta \sinh \phi - \sinh \Delta \cos \Theta \cosh \phi) \Lambda_{V_1} +$$

$$+ \frac{1}{\sinh \phi} \sinh \Delta \cos \Theta \Lambda_{V_1^*} + \sinh \Delta \sin \Theta a_{V_3}$$  (25)

is obtained.

Equation (25) may be also written by the angles $\Delta_i$ and $W_1$ as follows:

$$\Lambda_V = \Lambda_{V_1} \cosh W_2 \frac{\sinh \phi}{\sinh \phi} + \Lambda_{V_1^*} \cosh \Delta_2 \frac{\sinh \phi}{\sinh \phi} + \cosh \Delta_3 a_{V_3}.$$  (26)

If dual angle of pitch $\Lambda_V$ is expressed according to the frame $D^*$ from equations (11) and (19) some results are obtained. Thus, from equation (26) we have the following theorem.

**Theorem 6.** There is the relation (26) in the dual Lorentzian space $\mathbb{D}_3^1$, during the one-parameter dual closed Lorentzian spherical motion $K/K_1$, among the dual angle of pitch of the timelike closed ruled surface drawn by the timelike vector $\vec{V}$ which moves along the timelike closed ruled surfaces ($\vec{V}_1$) and ($\vec{V}_1^*$), the dual angle of pitch of the timelike closed ruled surfaces ($\vec{V}_1$), ($\vec{V}_1^*$) and closed Lorentzian spherical area bounded on the fixed sphere by the indicatrix of the vector $\vec{V}_3$. 

Now, let us consider some special cases. If the timelike vectors $\vec{V}_1$ and $\vec{V}^*_1$ draw the same ruled surface, then $\Lambda_{V_1} = \Lambda_{V^*_1}$.

In this case, we have only related to the moving along the timelike closed ruled surface ($\vec{V}_1$). Therefore, from equations (25) and (26) it is obtained in the following results:

$$\Lambda_V = \frac{\cosh W_2 + \cosh \Delta_2}{\sinh \phi} \Lambda_{V_1} + \cosh \Delta a_{V_3}. \quad (27)$$

Thus, we have the following theorem.

**Theorem 7.** During the one-parameter dual Lorentzian closed motion $K/K_1$, there is the relation (27) in the dual Lorentzian 3-space $\mathbb{D}^3_1$, among the dual angle of pitch of the timelike closed ruled surface drawn by the time-like vector $\vec{V}$ which is fixed in the frame $D$, the angle of pitch of the timelike closed ruled surface drawn by the timelike vector $\vec{V}_1$ and timelike closed dual Lorentzian spherical area bounded on the fixed dual sphere $K_1$ by the indicatrix of the vector $\vec{V}_3$.

Considering all the case, we choose fixed the timelike vector $\vec{V}$ according to frames $D$ and $D^*$. Now, let us take the timelike vectors $\vec{V}_1$, $\vec{V}^*_1$ and $\vec{V}$ on a great Lorentzian circle of the moving unit dual Lorentzian sphere $K$. In this case, $\Theta = 0$.

Thus,

$$< \vec{V}, \vec{V}_3 > = 0 \quad (28)$$

by equation (7), from equation (14)

$$\cosh W = \cosh \Delta \cosh \phi + \sinh \Delta \sinh \phi = \cosh(\Delta + \phi) \quad (29)$$

or $\Delta = W - \phi$ is obtained.

In this case, during the one-parameter dual Lorentzian closed motion $K/K_1$ we again study the motion of the timelike vector $\vec{V}$ which moves along the timelike closed ruled surfaces ($\vec{V}_1$) and ($\vec{V}^*_1$). When $\Theta = 0$, from equations (10) and (11) for the timelike vector $\vec{V}$, we have

$$\vec{V} = \cosh \Delta \vec{V}_1 + \sinh \Delta \vec{V}_2, \quad (30)$$

or

$$\vec{V} = \cosh(\Delta + \phi) \vec{V}^*_1 + \sinh(\Delta + \phi) \vec{V}^*_2. \quad (31)$$
Thus, from equations (25), (31) and \( \Theta = 0 \) the following result is obtained:

\[
\Lambda_V = \frac{\sinh(\Delta + \phi)}{\sinh \phi} \Lambda_{V_1} + \frac{\sinh \Delta}{\sinh \phi} \Lambda_{V_1}^*.
\]  

(32)

Thus, the following theorem is given.

**Theorem 8.** During the one-parameter dual closed Lorentzian spherical motion related to the closed dual Lorentzian indicatrix of the timelike closed ruled surface \((\vec{V}_1^*)\), there is the relation (32) in the Lorentzian 3-space \(\mathbb{D}_1^3\), among the dual angle of pitch of the timelike closed ruled surface drawn by the timelike vector \(\vec{V}\) which is fixed in the frame \(D\), the angles of pitch of the timelike closed ruled surfaces drawn by the timelike vectors \(\vec{V}_1\) and \(\vec{V}_1^*\), respectively.

Separating real and dual parts of equation (32) we find

\[
\lambda_V = \frac{\sinh(\varphi + \alpha)}{\sinh \varphi} \lambda_{V_1} + \frac{\sinh \alpha}{\sinh \varphi} \lambda_{V_1}^*;
\]

(33)

and

\[
l_V = l_{V_1} \frac{\sinh(\varphi + \alpha)}{\sinh \varphi} + l_{V_1}^* \frac{\sinh \alpha}{\sinh \varphi} + \varphi^* \lambda_V \frac{\cosh \varphi}{\sinh \varphi} - \alpha^* \lambda_{V_1} \frac{\cosh \alpha}{\sinh \varphi} - \lambda_{V_1} (\varphi^* + \alpha^*) \frac{\cosh(\varphi + \alpha)}{\sinh \varphi}.
\]

(34)

Thus, we give the following theorem in the line space \(\mathbb{R}_1^3\).

**Theorem 9.** During the one-parameter Lorentzian closed spatial motion \(H/H_1\) which corresponds to the one-parameter dual Lorentzian spherical closed motion \(K/K_1\) related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface \((\vec{V}_1^*)\), there are the relations (33) and (34) in the line space \(\mathbb{R}_1^3\), among the real angles of pitch and pitches of the timelike closed ruled surfaces drawn in \(H_1\) by the timelike line \(\vec{V}\) which is fixed in \(H\) and the timelike lines \(\vec{V}_1\) and \(\vec{V}_1^*\), respectively.

Now, we suppose that the timelike closed ruled surfaces \((\vec{V}_1)\) and \((\vec{V}_1^*)\) are the same. In this case, from equation (32) we get

\[
\Lambda_V = \frac{\sinh(\Delta + \phi) + \sinh \Delta}{\sinh \phi} \Lambda_{V_1};
\]

(35)
or
\[
\frac{\Lambda V}{\Lambda V_1} = \frac{\sinh(\Delta + \phi) + \sinh \Delta}{\sinh \phi}.
\] (36)

This result is important for us, because it is corresponds to Holditch Theorem for chosen Lorentzian spherical motion.

Therefore, the following theorem is given in dual Lorentzian space \( D_1^3 \).

**Theorem 10.** During the one-parameter closed Lorentzian spherical motion \( K/K_1 \) related to the closed dual Lorentzian indicatrix of the timelike closed ruled surface \( (\vec{V}_1^\ast) \), the ratio of the dual angle of pitch of the timelike closed ruled surface drawn on \( K_1 \) by the fixed timelike vector \( \vec{V} \) in the moving frame \( D \) to the dual angle of pitch of the timelike closed ruled surface drawn on \( K_1 \) by the timelike vector \( \vec{V}_1 \) is constant and independent of the motion.

Let us separate real and dual parts of equation (35). In this case, we get
\[
\lambda_V \sinh \varphi = \lambda_{V_1} [\sinh(\alpha + \varphi) + \sinh \alpha],
\] (37)
or
\[
\frac{\lambda_V}{\lambda_{V_1}} = \frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi},
\] (38)
and
\[
l_V = l_{V_1} \left[ \frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi} \right] - \\
- \lambda_{V_1} \left[ \frac{(\varphi^* + \alpha^*) \cosh(\alpha + \varphi) + \alpha^* \cosh \alpha}{\sinh \varphi} \right] + \lambda_V \varphi^* \frac{\cosh \varphi}{\sinh \varphi}.
\] (39)

Then, we have the following theorems in the line space \( \mathbb{R}_1^3 \).

**Theorem 11.** When the timelike closed ruled surfaces \( (\vec{V}_1) \) and \( (\vec{V}_1^\ast) \) are the same, during the one-parameter Lorentzian closed spatial motion \( H/H_1 \) which corresponds to the dual Lorentzian spherical closed motion \( K/K_1 \) related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface \( (\vec{V}_1 = \vec{V}_1^\ast) \), the ratio of the real angle of pitch of the timelike closed ruled surface drawn on \( H_1 \) by the fixed timelike vector \( \vec{V} \) in the moving frame \( D \) to the real angle of pitch of the timelike closed ruled surface drawn on \( H_1 \) by the timelike vector \( \vec{V}_1 \) is constant and independent of the Lorentzian motion and equal to the
\[
\frac{\lambda_V}{\lambda_{V_1}} = \frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi}.
\]
Theorem 12. When the timelike closed ruled surfaces $(\vec{V}_1)$ and $(\vec{V}^*_1)$ are the same, during the one-parameter Lorentzian closed spatial motion $H/H_1$ which corresponds to the dual Lorentzian spherical closed motion $K/K_1$ related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface $(\vec{V}_1 = \vec{V}^*_1)$, there is the relation (39), among the pitches and the real angles of pitch of the timelike closed ruled surfaces drawn in $H_1$ by the timelike line $\vec{V}$ which is fixed in $H$ and the timelike line $\vec{V}^*_1$.

References


