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ON THE GEOMETRY OF CLOSED TIMELIKE RULED SURFACES IN DUAL LORENTZIAN SPACE

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Abstract: In this paper, a dual timelike curve $c(t)$ which is a Lorentzian spherical indicatrix of a timelike closed ruled surface $\left(\overrightarrow{V}^*_1(t)\right)$ with a real parameter t, two frames $D^*(\overrightarrow{V}^*_1, \overrightarrow{V}^*_2, \overrightarrow{V}^*_3)$ related to the timelike closed ruled surface $\left(\overrightarrow{V}^*_1(t)\right)$ and $D(\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3)$ which moves respect to D^* and related to a timelike closed ruled surface drawn by a timelike vector $\overrightarrow{V}_1(t)$, and a timelike vector $\overrightarrow{V}$ which is fixed in the frame D are considered; and dual integral invariants of the timelike closed ruled surfaces which correspond to the dual timelike closed curves drawn by the vectors $\overrightarrow{V}$, $\overrightarrow{V}_1$ and $\overrightarrow{V}^*_1$ are studied; and it is found same relations among the dual integral invariants of the timelike closed ruled surfaces which correspond to the dual timelike closed curves drawn by the timelike vectors $\overrightarrow{V}$, $\overrightarrow{V}_1$ and $\overrightarrow{V}^*_1$. In addition, these results are carried to the Lorentzian line space $\mathbb{R}^3_1$ and give some theorems by means of Study's mapping.

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1. Introduction

The geometry of ruled surface is very important in the study of kinematics or spatial mechanism, surface design, manufacturing technology, robotic research in $\mathbb{R}^3$, [4,5,6,7,8,22]. It is well known, from [10] that a V_1- closed ruled surface generated by V_1- oriented line of rigid body have two real integral invariants; the real pitch l_{V_1} and the real angle of λ_{V_1}.

In recent years, by using the real integral invariants of closed ruled surfaces, many investigations are done, [3,9,23]. In addition, by using Lorentzian metric, ruled surfaces in Lorentz space have been studied [1,11,12,15,16,20].

In this paper, a dual closed spherical motion given by [24] is considered in the dual Lorentzian 3$-$space $\mathbb{D}^3_1$. During this motion, same relations among the dual integral invariants of the timelike ruled surfaces drawn by the timelike vectors $\overrightarrow{V}$, $\overrightarrow{V}_1$ and $\overrightarrow{V}^*_1$ are given. In addition, separating real and dual parts of these dual integral invariants, some results and some theorems related to Holditch Theorem are given in the Lorentzian 3-space $\mathbb{R}^3_1$ by Study's mapping.

2. Basic Concepts

We recall some fundamental concepts of the subject. If a and a^* are real numbers and $\epsilon^2 = 0$, the combination $A = a + \epsilon a^*$ is called a dual number, where ϵ is dual unit. The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$.

Let us consider Lorentzian 3$-$space $\mathbb{R}^3_1 = [\mathbb{R}^3, (-,+,+)]$ and the Lorentzian inner product of $\overrightarrow{A} = (a_1, a_2, a_3)$ and $\overrightarrow{B} = (b_1, b_2, b_3) \in \mathbb{R}^3$ be $< \overrightarrow{A}, \overrightarrow{B} >= -a_1b_1 + a_2b_2 + a_3b_3$.

The vector $\overrightarrow{A} = (a_1, a_2, a_3) \in \mathbb{R}^3_1$ is said to be timelike if $< \overrightarrow{A}, \overrightarrow{A} >< 0$, spacelike if $< \overrightarrow{A}, \overrightarrow{A} >> 0$ or $\overrightarrow{A} = 0$ and lightlike (null) if $< \overrightarrow{A}, \overrightarrow{A} >= 0$ and $\overrightarrow{A} \neq \overrightarrow{0}$.

Consider moving unit dual Lorentzian sphere K generated by a dual orthonormal system $(0; \overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3)$ and fixed unit dual Lorentzian sphere generated by a dual orthonormal system $(0; \overrightarrow{V}^*_1, \overrightarrow{V}^*_2, \overrightarrow{V}^*_3)$.

Definition 1. Consider a non-singular linear transformation between two dual orthonormal coordinate systems linked to the unit dual Lorentzian spheres K (moving) and K_1 (fixed), respectively. In the case that the corresponding matrix is an orthogonal matrix whose elements are differentiable dual functions of a real parameter t, the transformation is called a one-parameter dual Lorentzian spherical motion and denoted by K/K_1.

Theorem 2. ([19]) *There exists one-to-one correspondence between directed timelike (resp. spacelike) lines of $\mathbb{R}^3_1$ and an ordered pair of vectors $(\overrightarrow{a}, \overrightarrow{a}^*)$ such that $<\overrightarrow{a}, \overrightarrow{a}>= -1$ (resp. $<\overrightarrow{a}, \overrightarrow{a}>= 1$) and $<\overrightarrow{a}, \overrightarrow{a}^*>= 0$.*

Definition 3. A timelike closed ruled surface (X) is one-to-one correspondence with a dual closed curve on the Lorentzian sphere, [18].

Let $\left\{\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3\right\}$ and $\left\{\overrightarrow{V}^*_1, \overrightarrow{V}^*_2, \overrightarrow{V}^*_3\right\}$ be two right-handed sets of orthonormal unit dual vectors that are rigidly linked to the Lorentzian dual spheres K and K_1, and denoted by

$$\overrightarrow{V} = \begin{pmatrix} \overrightarrow{V}_1 \\ \overrightarrow{V}_2 \\ \overrightarrow{V}_3 \end{pmatrix}, \qquad \overrightarrow{V}^* = \begin{pmatrix} \overrightarrow{V}^*_1 \\ \overrightarrow{V}^*_2 \\ \overrightarrow{V}^*_3 \end{pmatrix}, \tag{1}$$

respectively. Between frames $\overrightarrow{V}$ and $\overrightarrow{V}^*$, the relation

$$\Omega = \begin{pmatrix} 0 & W_1^2 & W_1^3 \\ W_1^2 & 0 & -W_2^3 \\ W_1^3 & W_2^3 & 0 \end{pmatrix} \tag{2}$$

is given by [13]. We assume that $\overrightarrow{V}_3, \overrightarrow{V}_2, \overrightarrow{V}^*_3$ and $\overrightarrow{V}^*_2$ are spacelike dual vectors, $\overrightarrow{V}_1$ and $\overrightarrow{V}^*_1$ are timelike dual vectors in $\mathbb{D}^3_1$.

Definition 4. Let $\overrightarrow{\psi}$ be a dual pfaf vector during the dual motion K/K_1. The dual vector $\overrightarrow{D}$ is defined by

$$\overrightarrow{D} = \oint \overrightarrow{\psi} = \oint (W_2^3\overrightarrow{V}_1 + W_1^3\overrightarrow{V}_2 - W_1^2\overrightarrow{V}_3) = \overrightarrow{d} + \epsilon \overrightarrow{d}^* \tag{3}$$

is called dual Steiner vector of the dual motion K/K_1, [13].

Definition 5. The dual angle of pitch of a timelike closed ruled surface $(X(t))$ is given by [17] as

$$\Lambda_x = - <\overrightarrow{D}, \overrightarrow{X}> . \tag{4}$$

3. Dual Angles of Pitch of Closed Ruled Surfaces Drawn on the Fixed Dual Sphere K_1 by Vectors $\vec{V}, \vec{V}_1$, and $\vec{V}_1^*$

Let us choose a fixed dual point $\overrightarrow{V}_1^*$ which is on the unit dual moving Lorentzian sphere K. During the one-parameter dual Lorentzian spherical closed motion K/K_1, for every t, there is a constant dual angle ϕ between the vector $\overrightarrow{V}_1^*$ and unit dual vector $\overrightarrow{V}_1$ of the moving Lorentzian sphere K, i.e., let us choose

$$< \overrightarrow{V}_1, \overrightarrow{V}_1^* >= \cosh\phi. \tag{5}$$

During the dual closed Lorentzian spherical motion K/K_1, $\overrightarrow{V}_1$ and $\overrightarrow{V}_1^*$ draw two closed spherical curves on the unit dual fixed Lorentzian sphere K_1. These curves correspond to two closed ruled surfaces in the fixed line space H_1 by Study's mapping. Let us denote the closed ruled surfaces drawn by the vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}_1^*$ by $(\overrightarrow{V}_1)$ and $(\overrightarrow{V}_1^*)$, respectively.

Consider $\left\{\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3\right\}$ and $\left\{\overrightarrow{V}_1^*, \overrightarrow{V}_2^*, \overrightarrow{V}_3^*\right\}$ two sets of orthonormal unit dual vectors that are rigidly linked to the Lorentzian spheres K and K_1, and denoted by

$$D(\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3), \quad D^*(\overrightarrow{V}_1^*, \overrightarrow{V}_2^*, \overrightarrow{V}_3^*) \tag{6}$$

respectively, where frames D and D^* are taken form by the vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}_1^*$, respectively.

These two frames are rigidly linked to D and D^* by

$$\begin{aligned} \overrightarrow{V}_3 &= \tfrac{\overrightarrow{V}_1 \times \overrightarrow{V}_1^*}{\sinh\phi}, \\ \overrightarrow{V}_2 &= \overrightarrow{V}_3 \times \overrightarrow{V}_1, \\ \overrightarrow{V}_3 &= -\overrightarrow{V}_3^*, \\ \overrightarrow{V}_2^* &= \overrightarrow{V}_3^* \times \overrightarrow{V}_1^*. \end{aligned} \tag{7}$$

Thus, for the vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}_1^*$ we have

$$\begin{aligned} \overrightarrow{V}_1^* &= \cosh\phi\overrightarrow{V}_1 + \sinh\phi\overrightarrow{V}_2 \\ \overrightarrow{V}_2^* &= \sinh\phi\overrightarrow{V}_1 + \cosh\phi\overrightarrow{V}_2 \\ \overrightarrow{V}_3^* &= -\overrightarrow{V}_3 \end{aligned} \ , \tag{8}$$

or

$$\begin{aligned} \overrightarrow{V}_1 &= \cosh\phi\overrightarrow{V}_1^* + \sinh\phi\overrightarrow{V}_2^* \\ \overrightarrow{V}_2 &= \sinh\phi\overrightarrow{V}_1^* + \cosh\phi\overrightarrow{V}_2^* \\ \overrightarrow{V}_3 &= -\overrightarrow{V}_3^*, \qquad (\phi = \varphi + \epsilon\varphi^*) \end{aligned} \ . \tag{9}$$

Let us denote the dual hyperbolic angle of the vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}$ by Δ and the angle of the vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}^*_1$ by ϕ.

Let us consider an unit dual timelike vector $\overrightarrow{V}$ which is fixed in the orthonormal frame D and expressed as follows:

$$\overrightarrow{V} = \cosh\Delta\overrightarrow{V}_1 + \sinh\Delta\cos\Theta\overrightarrow{V}_2 + \sinh\Delta\sin\Theta\overrightarrow{V}_3, \tag{10}$$

$(\Delta = \alpha + \epsilon\alpha^*, \Theta = \theta + \epsilon\theta^*)$, where $\overrightarrow{V}_1$ is timelike.

In addition, unit dual timelike vector $\overrightarrow{V}$ with respect to dual orthonormal frame D^* by equations (9) and (10) also is written as follows:

$$\begin{aligned}\overrightarrow{V} = &(\cosh\Delta\cosh\phi + \sinh\Delta\sinh\phi\cos\Theta)\overrightarrow{V}^*_1\\ &+(\cosh\Delta\sinh\phi + \sinh\Delta\cosh\phi\cos\Theta)\overrightarrow{V}^*_2 - \sinh\Delta\sin\Theta\overrightarrow{V}_3.\end{aligned} \tag{11}$$

By means of equation (10)

$$< \overrightarrow{V}_1, \overrightarrow{V} >= \cosh\Delta. \tag{12}$$

In the same way

$$< \overrightarrow{V}^*_1, \overrightarrow{V} >= \cosh W. \tag{13}$$

Equation (13) by means of equation(11) is written as follows:

$$\cosh W = \cosh\Delta\cosh\phi + \sinh\Delta\sinh\phi\cos\Theta.$$

For $\Theta = 0$, from equation (13)

$$\cosh W = \cosh\Delta\cosh\phi + \sinh\Delta\sinh\phi = \cosh(\Delta + \phi), \tag{14}$$

or $W = \Delta + \phi$ is obtained.

In addition, constant dual hyperbolic angles Δ_i and W_i which are among the unit dual vectors $(\overrightarrow{V}_i, \overrightarrow{V})$ and $(\overrightarrow{V}^*_i, \overrightarrow{V})$ are given as follows:

$$\cosh\Delta_i =< \overrightarrow{V}_i, \overrightarrow{V} >$$

and

$$\cosh W_i =< \overrightarrow{V}^*_i, \overrightarrow{V} >, \quad (i = 1, 2, 3), \tag{15}$$

respectively.

From equations (10) and (11), the relations among constant angles Δ_i and W_i as depend on the angles Δ, ϕ and Θ are obtained.

Thus, during the dual Lorentzian motion K/K_1 we may consider the timelike closed ruled surface drawn by the timelike vector $\overrightarrow{V}$ which is fixed in the orthonormal frames D and D^*. This timelike closed ruled surface is expressed as $(\overrightarrow{V})$.

Dual angle of pitch of the timelike closed ruled surface $(\overrightarrow{V})$ by means of equations (3) and (4) is written as follows:

$$\Lambda_V = - < \overrightarrow{D}, \overrightarrow{V} > . \tag{16}$$

Thus, from equations (10) and (16)

$$\Lambda_V = \cosh \Delta \oint W_2^3 - \sinh \Delta \cos \Theta \oint W_1^3 + \sinh \Delta \sin \Theta \oint W_1^2 \tag{17}$$

is obtained.

The integral $\oint W_2^3$ is the dual angle of pitch of the timelike closed ruled surface $(\overrightarrow{V}_1)$. In this case

$$\Lambda_{V_1} = \oint W_2^3 = - < \overrightarrow{D}, \overrightarrow{V}_1 > \tag{18}$$

may be written.

In addition, because of this also stated the unit dual timelike vector $\overrightarrow{V}$ according to the frame D^*, in the same way

$$\Lambda_V = - < \overrightarrow{D}^*, \overrightarrow{V}^* > \tag{19}$$

may be written, where $\overrightarrow{D}^*$ is given as follows:

$$\overrightarrow{D}^* = \oint [(W_2^3)^* \overrightarrow{V}_1 + (W_1^3)^* \overrightarrow{V}_2^* - (W_1^2)^* \overrightarrow{V}_3^*]. \tag{20}$$

The integral $\oint (W_1^3)$ may be consider as a function of the dual angle of pitch Λ_{V_1} and $\Lambda_{V_1^*}$ of the timelike closed ruled surfaces $(\overrightarrow{V}_1)$ and $(\overrightarrow{V}_1^*)$, respectively. For this case, we consider the dual angle of pitch of the closed ruled surface $(\overrightarrow{V}_1^*)$. Thus, from the equations (18) and (16), $\Lambda_{V_1^*}$ is written as follows:

$$\Lambda_{V_1^*} = - < \overrightarrow{D}^*, \cosh \phi \overrightarrow{V}_1 + \sinh \phi \overrightarrow{V}_2 >,$$

or

$$\Lambda_{V_1^*} = \cosh \phi \oint W_2^3 - \sinh \phi \oint W_1^3. \tag{21}$$

Thus, equation (21) is written as follows:

$$\Lambda_{V_1^*} = \Lambda_{V_1} \cosh\phi - \sinh\phi \oint W_1^3. \tag{22}$$

From equation (22), for the integral $-\oint W_1^3$ we have

$$-\oint W_1^3 = \frac{1}{\sinh\phi}(\Lambda_{V_1^*} - \Lambda_{V_1}\cosh\phi). \tag{23}$$

For meaning of integral $\oint W_1^2$ we use equalities in the following:

$$\oint W_1^2 = \oint_{\partial G_3} [d, W_1^2] = \oint_{\partial G_3} [W_1^3, W_1^2] = a_{V_3} \tag{24}$$

is obtained, where ∂G_3 is the dual Lorentzian spherical area bounded on the fixed dual Lorentzian sphere by the dual Lorentzian spherical indicatrix of the spacelike vector $\overrightarrow{V}_3$ [19]. Thus, from equations (17), (18), (23) and (24)

$$\Lambda_V = \Lambda_{V_1}\cosh\Delta + \frac{\sinh\Delta\cos\Theta}{\sinh\phi}(\Lambda_{V_1^*} - \Lambda_{V_1}\cosh\phi) + \sinh\Delta\sin\Theta a_{V_3},$$

or

$$\begin{aligned}\Lambda_V = &\frac{1}{\sinh\phi}(\cosh\Delta\sinh\phi - \sinh\Delta\cos\Theta\cosh\phi)\Lambda_{V_1} + \\ &+\frac{1}{\sinh\phi}\sinh\Delta\cos\Theta\Lambda_{V_1^*} + \sinh\Delta\sin\Theta a_{V_3}\end{aligned} \tag{25}$$

is obtained.

Equation (25) may be also written by the angles Δ_i and W_i as follows:

$$\Lambda_V = \Lambda_{V_1}\frac{\cosh W_2}{\sinh\phi} + \Lambda_{V_1^*}\frac{\cosh\Delta_2}{\sinh\phi} + \cosh\Delta_3 a_{V_3}. \tag{26}$$

If dual angle of pitch Λ_V is expressed according to the frame D^* from equations (11) and (19) some results are obtained. Thus, from equation (26) we have the following theorem.

Theorem 6. *There is the relation* (26) *in the dual Lorentzian space* $\mathbb{D}_1^3$*, during the one-parameter dual closed Lorentzian spherical motion* K/K_1*, among the dual angle of pitch of the timelike closed ruled surface drawn by the timelike vector* $\overrightarrow{V}$ *which moves along the timelike closed ruled surfaces* $(\overrightarrow{V}_1)$ *and* $(\overrightarrow{V}_1^*)$*, the dual angle of pitch of the timelike closed ruled surfaces* $(\overrightarrow{V}_1)$*,* $(\overrightarrow{V}_1^*)$ *and closed Lorentzian spherical area bounded on the fixed sphere by the indicatrix of the vector* $\overrightarrow{V}_3$*.*

Now, let us consider some special cases. If the timelike vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}_1^*$ draw the same ruled surface, then $\Lambda_{V_1} = \Lambda_{V_1^*}$.

In this case, we have only related to the moving along the timelike closed ruled surface $(\overrightarrow{V}_1)$. Therefore, from equations (25) and (26) it is obtained in the following results:

$$\Lambda_V = \frac{\cosh W_2 + \cosh \Delta_2}{\sinh \phi} \Lambda_{V_1} + \cosh \Delta_3 a_{V_3}. \tag{27}$$

Thus, we have the following theorem.

Theorem 7. *During the one-parameter dual Lorentzian closed motion K/K_1, there is the relation (27) in the dual Lorentzian $3-$ space $\mathbb{D}_1^3$, among the dual angle of pitch of the timelike closed ruled surface drawn by the timelike vector $\overrightarrow{V}$ which is fixed in the frame D, the angle of pitch of the timelike closed ruled surface drawn by the timelike vector $\overrightarrow{V}_1$ and timelike closed dual Lorentzian spherical area bounded on the fixed dual sphere K_1 by the indicatrix of the vector $\overrightarrow{V}_3$.*

Considering all the case, we choose fixed the timelike vector $\overrightarrow{V}$ according to frames D and D^*. Now, let us take the timelike vectors $\overrightarrow{V}_1$, $\overrightarrow{V}_1^*$ and $\overrightarrow{V}$ on a great Lorentzian circle of the moving unit dual Lorentzian sphere K. In this case, $\Theta = 0$.

Thus,

$$< \overrightarrow{V}, \overrightarrow{V}_3 >= 0 \tag{28}$$

by equation (7), from equation (14)

$$\cosh W = \cosh \Delta \cosh \phi + \sinh \Delta \sinh \phi = \cosh(\Delta + \phi) \tag{29}$$

or $\Delta = W - \phi$ is obtained.

In this case, during the one-parameter dual Lorentzian closed motion K/K_1 we again study the motion of the timelike vector $\overrightarrow{V}$ which moves along the timelike closed ruled surfaces $(\overrightarrow{V}_1)$ and $(\overrightarrow{V}_1^*)$. When $\Theta = 0$, from equations (10) and (11) for the timelike vector $\overrightarrow{V}$, we have

$$\overrightarrow{V} = \cosh \Delta \overrightarrow{V}_1 + \sinh \Delta \overrightarrow{V}_2, \tag{30}$$

or

$$\overrightarrow{V} = \cosh(\Delta + \phi) \overrightarrow{V}_1^* + \sinh(\Delta + \phi) \overrightarrow{V}_2^*. \tag{31}$$

Thus, from equations (25), (31) and $\Theta = 0$ the following result is obtained:

$$\Lambda_V = \frac{\sinh(\Delta + \phi)}{\sinh \phi}\Lambda_{V_1} + \frac{\sinh \Delta}{\sinh \phi}\Lambda_{V_1^*}. \tag{32}$$

Thus, the following theorem is given.

Theorem 8. *During the one-parameter dual closed Lorentzian spherical motion related to the closed dual Lorentzian indicatrix of the timelike closed ruled surface* $(\overrightarrow{V}^*_1)$*, there is the relation* (32) *in the Lorentzian* $3-$ *space* $\mathbb{D}^3_1$*, among the dual angle of pitch of the timelike closed ruled surface drawn by the timelike vector* $\overrightarrow{V}$ *which is fixed in the frame* D*, the angles of pitch of the timelike closed ruled surfaces drawn by the timelike vectors* $\overrightarrow{V}_1$ *and* $\overrightarrow{V}^*_1$*, respectively.*

Separating real and dual parts of equation (32) we find

$$\lambda_V = \frac{\sinh(\varphi + \alpha)}{\sinh \varphi}\lambda_{V_1} + \frac{\sinh \alpha}{\sinh \varphi}\lambda_{V_1^*} \tag{33}$$

and

$$\begin{aligned} l_V = l_{V_1}\frac{\sinh(\varphi + \alpha)}{\sinh \varphi} + l_{V_1^*}\frac{\sinh \alpha}{\sinh \varphi} + \varphi^*\lambda_V\frac{\cosh \varphi}{\sinh \varphi} - \alpha^*\lambda_{V_1^*}\frac{\cosh \alpha}{\sinh \varphi} \\ -\lambda_{V_1}(\varphi^* + \alpha^*)\frac{\cosh(\varphi + \alpha)}{\sinh \varphi}. \end{aligned} \tag{34}$$

Thus, we give the following theorem in the line space $\mathbb{R}^3_1$.

Theorem 9. *During the one-parameter Lorentzian closed spatial motion* H/H_1 *which corresponds to the one-parameter dual Lorentzian spherical closed motion* K/K_1 *related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface* $(\overrightarrow{V}^*_1)$*, there are the relations* (33) *and* (34) *in the line space* $\mathbb{R}^3_1$*, among the real angles of pitch and pitches of the timelike closed ruled surfaces drawn in* H_1 *by the timelike line* $\overrightarrow{V}$ *which is fixed in* H *and the timelike lines* $\overrightarrow{V}_1$ *and* $\overrightarrow{V}^*_1$*, respectively.*

Now, we suppose that the timelike closed ruled surfaces $(\overrightarrow{V}_1)$ and $(\overrightarrow{V}^*_1)$ are the same. In this case, from equation (32) we get

$$\Lambda_V = \frac{\sinh(\Delta + \phi) + \sinh \Delta}{\sinh \phi}\Lambda_{V_1}, \tag{35}$$

or

$$\frac{\Lambda_V}{\Lambda_{V_1}} = \frac{\sinh(\Delta + \phi) + \sinh \Delta}{\sinh \phi} \tag{36}$$

This result is important for us, because it is corresponds to Holditch Theorem for chosen Lorentzian spherical motion.

Therefore, the following theorem is given in dual Lorentzian space $\mathbb{D}^3_1$.

Theorem 10. *During the one-parameter closed Lorentzian spherical motion K/K_1 related to the closed dual Lorentzian indicatrix of the timelike closed ruled surface $(\overrightarrow{V}^*_1)$, the ratio of the dual angle of pitch of the timelike closed ruled surface drawn on K_1 by the fixed timelike vector $\overrightarrow{V}$ in the moving frame D to the dual angle of pitch of the timelike closed ruled surface drawn on K_1 by the timelike vector $\overrightarrow{V}_1$ is constant and independent of the motion.*

Let us separate real and dual parts of equation (35). In this case, we get

$$\lambda_V \sinh \varphi = \lambda_{V_1}[\sinh(\alpha + \varphi) + \sinh \alpha], \tag{37}$$

or

$$\frac{\lambda_V}{\lambda_{V_1}} = \frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi}, \tag{38}$$

and

$$\begin{aligned} l_V = l_{V_1}[&\frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi}] - \\ -\lambda_{V_1}[&\frac{(\varphi^* + \alpha^*)\cosh(\alpha + \varphi) + \alpha^* \cosh \alpha}{\sinh \varphi}] + \lambda_V \varphi^* \frac{\cosh \varphi}{\sinh \varphi}. \end{aligned} \tag{39}$$

Then, we have the following theorems in the line space $\mathbb{R}^3_1$.

Theorem 11. *When the timelike closed ruled surfaces $(\overrightarrow{V}_1)$ and $(\overrightarrow{V}^*_1)$ are the same, during the one-parameter Lorentzian closed spatial motion H/H_1 which corresponds to the dual Lorentzian spherical closed motion K/K_1 related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface $(\overrightarrow{V}_1 = \overrightarrow{V}^*_1)$, the ratio of the real angle of pitch of the timelike closed ruled surface drawn on H_1 by the fixed timelike vector $\overrightarrow{V}$ in the moving frame D to the real angle of pitch of the timelike closed ruled surface drawn on H_1 by the timelike vector $\overrightarrow{V}_1$ is constant and independent of the Lorentzian motion and equal to the*

$$\frac{\lambda_V}{\lambda_{V_1}} = \frac{\sinh(\alpha + \varphi) + \sinh \alpha}{\sinh \varphi}.$$

Theorem 12. *When the timelike closed ruled surfaces* $(\overrightarrow{V}_1)$ *and* $(\overrightarrow{V}^*_1)$ *are the same, during the one-parameter Lorentzian closed spatial motion* H/H_1 *which corresponds to the dual Lorentzian spherical closed motion* K/K_1 *related to the closed dual Lorentzian spherical indicatrix of the timelike closed ruled surface* $(\overrightarrow{V}_1 = \overrightarrow{V}^*_1)$*, there is the relation* (39)*, among the pitches and the real angles of pitch of the timelike closed ruled surfaces drawn in* H_1 *by the timelike line* $\overrightarrow{V}$ *which is fixed in* H *and the timelike line* $\overrightarrow{V}^*_1$*.*

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