GENERALIZATION OF THE WIMAN-VALIRON METHOD FOR FRACTIONAL DERIVATIVES

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Abstract: We generalize the Wiman-Valiron method for fractional derivatives proving that

\[ |z|^q D^q f(z) \sim (\nu(r,f))^q f(z) \]

holds in a neighborhood of a maximum modulus point outside an exceptional set of values of \(|z|\) as \(|z| \to \infty\), where \(D^q\) is the Riemann-Liouville fractional derivative of order \(q > 0\), \(\nu(r,f)\) is the central index of the Taylor representation of \(f\). We use this result to find the precise value for the order of growth of solutions of a fractional differential equation.

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1. Introduction

Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\theta} \]

be a transcendental entire function. For \(r \in [0, +\infty)\) we denote \(M(r,f) = \)
max\{|f(z)| : |z| = r\}, and let \(\mu(r, f) = \max\{|a_n|r^n : n \geq 0\}\) be the maximum term and \(\nu(r, f) = \max\{n \geq 0 : |a_n|r^n = \mu(r, f)\}\) be the central index of the series (1).

The theory, initiated by A. Wiman ([14, 15]) and developed by many other mathematicians such as G. Valiron, J. Clunie, T. Kövari, describes the local behavior of \(f\) near a point \(z_r, |z_r| = r\), satisfying \(|f(z_r)| = M(r, f)\) in terms of the power series (1). A nice exposition is due to W.K. Hayman [4], where bibliographical references are given. The seminal result of the theory states that given \(q \in \mathbb{N}\) in a neighborhood of \(z_r\) one has

\[
f(z) \sim \left(\frac{z}{z_r}\right)^{\nu(r,f)} f(z_r), \quad f^{(q)}(z) \sim \left(\frac{\nu(r,f)}{z}\right)^q f(z)
\]

for \(r \in [1, \infty)\setminus E\) where \(E\) is a set of finite logarithmic measure, i.e. \(\int_{E \cap [1, \infty)} \frac{dx}{x} < \infty\). Another elegant approach not involving power series was proposed by A. Macintyre ([7]), who used \(K(r, f) := z_r f'(z_r)/f(z_r) = r(\log M(r, f))'\) instead of the central index. This approach was developed by Sh. Strelitz in his book [13], who proved counterparts of (2) with \(K(r, f)\) instead of \(\nu(r, f)\) for functions analytic in a strip or in the unit disc, and for Dirichlet series. On the other hand, the theory has been developed for Dirichlet series by M. Sheremeta and O. Skaskiv (see e.g. [9, 10, 12]). Recently, W. Bergweiler and others developed Macintyre’s approach for meromorphic functions having a direct tract in \(\mathbb{C}\) ([1]). Correlations (2) are very useful in studying differential equations. They allow to obtain sharp asymptotic estimates for the growth of solutions (see [16, 6, 1]). Counterparts of (2) for fractional values of \(q\) is unknown.

The aim of the paper is to obtain an analogue of the second relation of (2) for the Riemann-Liouville fractional derivatives.

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### 2. Generalization of the Wiman-Valiron Method for Fractional Derivatives

We start with the settings of the Wiman-Valiron theory.

Let \((\alpha_n)_{n=0}^\infty\) be a sequence of positive numbers such that \(\alpha_{n+1}/\alpha_n\) decreases with increasing \(n\). Let \((\varrho_n)\) be a sequence of numbers such that

\[
0 < \varrho_0 < \frac{\alpha_0}{\alpha_1}, \quad \frac{\alpha_{n-1}}{\alpha_n} < \varrho_n < \frac{\alpha_n}{\alpha_{n+1}} \quad (n \geq 1),
\]

so that \((\varrho_n)\) increases with increasing \(n\). We shall say that a value \(r\) is normal
(for the sequence \((a_n), (\alpha_n)\) and \((g_n)\)), if we have for some \(\nu\)

\[
|a_n| r^n \leq |a_\nu| r^\nu \frac{\alpha_n g^\nu_n}{\alpha_\nu g^\nu_\nu} \quad (n \geq 0).
\]

Let \(V\) be the class of positive continuous nondecreasing functions \(v\) on \([0, +\infty)\) such that \(\frac{x^2}{v(x) \ln v(x)}\) increases to \(+\infty\) on \(x \in [x_0; +\infty)\), \(x_0 > 0\), and \(\int_0^+ \frac{dx}{v(x)} < +\infty\). For example, the functions \(v(x) = x \ln^{\alpha+1} x\), \((x \geq e)\), \(\alpha > 0\), and \(v(x) = x^{\delta+1}\), \((x \geq 1)\), \(\delta \in (0, 1)\) belong to \(V\).

The main result of the Wiman-Valiron theory is formulated as follows.

**Theorem 1.** Let \(v \in V\) and \(\kappa(t) = 4\sqrt{v(t) \ln v(t)}\). Suppose that \(f\) is an entire function, a value \(r\) is normal and large enough, \(|z_0| = r\),

\[
|f(z_0)| \geq \eta M(r, f), \quad v^{-2}(\nu(r, f)) \leq \eta \leq 1,
\]

holds, and

\[
r \left(1 - \frac{1}{40\kappa(\nu)}\right) < \rho < r \left(1 + \frac{1}{40\kappa(\nu)}\right), \quad \nu = \nu(r, f).
\]

Then if \(q \in \mathbb{Z}_+\) we have for \(|z| = \rho\)

\[
\left(\frac{z}{\nu}\right)^q f^{(q)}(z) = f(z) + O\left(\frac{\kappa(\nu)}{\nu}\right) M(\rho, f).
\]

In particular, if \(\ln \rho - \ln r = o\left(\frac{1}{\kappa(\nu)}\right)\), then

\[
M(\rho, f^{(q)}) = \left(\frac{\nu}{\rho}\right)^q \left\{1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right\} M(\rho, f)
\]

\[
= (1 + o(1)) \left(\frac{\nu}{r}\right)^q M(r, f)
\]

as \(r \to +\infty\) outside a set of finite logarithmic measure.

We generalize the Wiman-Valiron method for fractional derivatives.

Let \(f \in L(0, a)\), \(a > 0\). The Riemann-Liouville fractional derivative of order \(\alpha > 0\) for \(f\) is defined as

\[
D^\alpha f(x) = \frac{d^n}{dx^n} \{t^{n-\alpha} f(x)\}, \quad \alpha \in (n-1, n], \quad n \in \mathbb{N},
\]
where

\[ I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \]

is the Riemann-Liouville fractional integral of order \( \alpha > 0 \) for \( f \), \( \Gamma(\alpha) \) is the Gamma function. In particular, if \( 0 < \alpha < 1 \), then

\[ D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)dt}{(x-t)^\alpha}. \]

The fractional derivative has the following property ([8]):

\[ D^\alpha x^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{\beta-\alpha-1}, \quad \alpha, \beta > 0. \] (3)

It follows from (3) that the fractional derivative for the entire function (1) is defined as

\[ |z|^\alpha D^\alpha f(z) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} z^n. \]

**Theorem 2.** Let \( v \in V \) and \( \kappa(t) = 4\sqrt{v(t) \ln v(t)} \). Suppose that \( f \) is an entire function, a value \( r \) is normal and large enough, \( |z_0| = r \),

\[ |f(z_0)| \geq \eta M(r, f), \quad v^{-2}(\nu(r, f)) \leq \eta \leq 1 \]

holds, and

\[ r \left(1 - \frac{1}{40\kappa(\nu)}\right) < \rho < r \left(1 + \frac{1}{40\kappa(\nu)}\right), \quad \nu = \nu(r, f). \]

Then if \( q > 0 \) we have for \( |z| = \rho \):

\[ \frac{\rho^n D^q f(z)}{\nu^q} = f(z) + O \left(\frac{\kappa(\nu)}{\nu}\right) M(\rho, f). \] (4)

In particular, if \( \ln \rho - \ln r = o \left(\frac{1}{\kappa(\nu)}\right) \), then

\[ M(\rho, D^q f(z)) = \left(\frac{\nu}{\rho}\right)^q \left(1 + O \left(\frac{\kappa(\nu)}{\nu}\right)\right) M(\rho, f) \]

\[ = (1 + o(1)) \left(\frac{\nu}{r}\right)^q M(r, f) \] (5)

as \( r \to +\infty \) outside a set of finite logarithmic measure.
For $\rho \in [0; +\infty)$ we set $\mu_r(\rho, f) = |a_n(r, f)| \rho^\nu(r, f)$.

To prove Theorem 2, we need the following statements.

**Lemma 1.** [11, Lemma 3.4], cf. [4, Lemma 2] Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t) \ln v(t)}$. Then we have for any fixed positive $q$ and for all $\rho$, $|\ln \rho - \ln r| \leq \frac{1}{\kappa(\nu)}$,

$$
\sum_{|n-\nu| > \kappa(\nu)} n^q |a_n| \rho^n = o\left(\frac{\nu^q \mu(\rho, \nu, f)}{v(\nu)^3}\right), \quad \nu = \nu(r, f),
$$

(6) as $r \to +\infty$ outside a set of finite logarithmic measure.

**Lemma 2.** [11, Lemma 3.5], cf. [4, Lemma 7] Suppose that $P$ is a polynomial of degree $m$ and $|P(z)| \leq M$ for $|z| \leq r$. Then for $R \geq r$ we have

$$
|P'(z)| \leq \frac{eMmR^{m-1}}{r^m}, \quad |z| < R.
$$

**Theorem 3.** [11, Lemma 3.7], cf. [4, Theorem 10] Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t) \ln v(t)}$. Suppose that $f$ is an entire function, a value $r$ is normal and enough large, $|z_0| = r$,

$$
|f(z_0)| \geq \eta M(r, f), \quad v^{-2}(\nu(r, f)) \leq \eta \leq 1.
$$

Then, if $z = z_0 e^\tau$, $|\tau| \leq \frac{\eta}{18\kappa(\nu)}$, $\nu = \nu(r, f)$, we have

$$
\ln \frac{f(z)}{f(z_0)} = (\nu(r, f) + \varphi_1)\tau + \varphi_2\tau^2 + \delta(\tau),
$$

where

$$
|\varphi_j| \leq 2, 2\left(\frac{18\kappa(\nu)}{\eta}\right)^j, \quad (j = 1, 2), \quad |\delta(\tau)| \leq 8, 8\left(\frac{18\kappa(\nu)\tau}{\eta}\right)^3.
$$

**Proof.** Let

$$
\nu_1 = \min\{n : |n-\nu| \leq \kappa(\nu)\}, \quad \nu_2 = \max\{n : |n-\nu| \leq \kappa(\nu)\}.
$$

We write

$$
f(z) = P(z)z^{\nu_1} + R(z),
$$

(7) where

$$
P(z) = \sum_{|n-\nu| \leq \kappa(\nu)} |a_n| z^{n-\nu_1}.
$$

(8)
Since \( \mu(r, \rho, f) \leq M(\rho, f) \), from Lemma 1 with \( q = 0 \) for all \( \rho \), \( |\ln \rho - \ln r| \leq \frac{1}{\kappa(\nu)} \), we obtain for \( |z| = \rho \)
\[
f(z) = P(z)z^{\nu_1} + o\left( \frac{\mu(r, \rho, f)}{v(\nu)^3} \right) = P(z)z^{\nu_1} + o\left( \frac{M(\rho, f)}{v(\nu)^3} \right),
\]
as \( r \to +\infty \) outside a set \( E \) of finite logarithmic measure.

In particular, from (9) with \( \rho = r \) we have
\[
|P(z)| \leq \frac{1.01 M(r, f)}{r^{\nu_1}} =: M^*(r), \quad |z| = r.
\]

We need the asymptotic representation for Gamma functions ([5])
\[
\frac{\Gamma(t + a)}{\Gamma(t + b)} = t^{a-b}\left(1 + O\left(\frac{1}{t}\right)\right), \quad t \to +\infty, \quad b, a \in \mathbb{R}.
\]

First we estimate the fractional derivative of order \( q \) for \( R(z) \). From (3) and Lemma 1 we deduce
\[
|\rho^q D^q R(z)| = \left| \rho^q \sum_{|n-\nu|>\kappa(\nu)} \frac{\Gamma(1+n)}{\Gamma(1+n-q)} a_n \rho^{n-q} e^{zg}\right|
\leq C \sum_{|n-\nu|>\kappa(\nu)} n^q |a_n| \rho^n = o\left( \frac{\nu^q \mu(r, \rho, f)}{v(\nu)^3} \right),
\]
where \( \nu = \nu(r, f) \), \( r \to +\infty \), \( r / \notin E \), and \( C = \sup\{2, \frac{\Gamma(n+1)}{\Gamma(n+1-q)} n^{-q}\} \).

Repeated application of Lemma 2 shows that for any \( q \in \mathbb{Z}_+ \) and \( |z| = \rho \)
\[
|P(q)(z)| \leq \left( \frac{6\kappa(\nu)}{r} \right)^q M^*(r).
\]

In fact,
\[
|P'(z)| \leq \frac{e M^*(r) 2\kappa(\nu) \rho^{\nu_2-\nu_1-1}}{r^{\nu_2-\nu_1}}
\leq \frac{2e M^*(r) \kappa(\nu)}{q} \left(1 + \frac{1}{40\kappa(\nu)}\right)^{2\kappa(\nu)} \leq \frac{6\kappa(\nu)}{r} M^*(r), \quad r \to +\infty
\]
and, similarly,
\[
|P(j)(z)| \leq \frac{e \max\{|P(j-1)(z)| : |z| \leq \rho\}(2\kappa(\nu) - j + 1) \rho^{\nu_2-\nu_1-j}}{r^{\nu_2-\nu_1-j+1}}
\]
\[ \leq \left( \frac{6\kappa(\nu)}{r} \right)^j M^*(r). \]

We need generalized Leibniz’s formula for fractional derivatives to estimate the first summand in (7). Let \( f(x) \) and \( g(x) \) be analytic functions on \([a, b]\), then ([8, p. 278])

\[ D^\alpha(f \cdot g) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} (D^{\alpha-k}f)g^{(k)}, \quad (14) \]

where \( \binom{\alpha}{k} = \frac{(-1)^k \alpha \Gamma(k - \alpha)}{\Gamma(1 - \alpha)\Gamma(k + 1)}. \)

It follows from (3) and (14) that

\[ \rho^q D^q(z^{\nu_1} P(z)) = \rho^q \sum_{m=0}^{+\infty} \binom{q}{m} D^{q-m}z^{\nu_1} P^{(m)}(z) \]

\[
= \sum_{m=0}^{\nu_2-\nu_1} \binom{q}{m} \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q + m)} z^{\nu_1} \rho^m P^{(m)}(z) \\
= \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} z^{\nu_1} \left( P(z) + \sum_{m=1}^{\nu_2-\nu_1} \binom{q}{m} \frac{\Gamma(\nu_1 + 1 - q)}{\Gamma(\nu_1 + 1 - q + m)} \rho^m P^{(m)}(z) \right). 
\]

We now estimate the second term in parentheses taking into account (13) and (11)

\[
\left| \sum_{m=1}^{2\kappa(\nu)} \binom{q}{m} \frac{\Gamma(\nu_1 + 1 - q)}{\Gamma(\nu_1 + 1 - q + m)} \rho^m P^{(m)}(z) \right| \leq \sum_{m=1}^{2\kappa(\nu)} \frac{q \Gamma(m - q) \Gamma(\nu_1 + 1 - q)}{\Gamma(1 - q)\Gamma(m + 1)\Gamma(\nu_1 + 1 - q + m)} \left( \frac{6\rho\kappa(\nu)}{r} \right)^m M^*(r) \\
\leq C(q) \sum_{m=1}^{2\kappa(\nu)} \frac{1}{m^{1+q}} \nu_1^{-m} C^m \kappa(\nu)^m M^*(r) \\
\leq C(q) \frac{\kappa(\nu)}{\nu} M^*(r) \sum_{m=1}^{2\kappa(\nu)} \frac{1}{m^{1+q}} = O\left( \frac{\kappa(\nu)}{\nu} \right) M^*(r). 
\]
Therefore, in view of (9) and the previous estimate we have

\[ \rho^q D^q (P(z) z^{\nu_1}) = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} z^{\nu_1} \left( P(z) + O \left( \frac{\kappa(\nu)}{\nu} M^*(r) \right) \right) \]

\[ = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} \left( f(z) + o \left( \frac{\mu(r, \rho, f)}{v(\nu)^3} \right) + O \left( \frac{\kappa(\nu)}{\nu} M^*(r) \rho^{\nu_1} \right) \right). \]  

(15)

Since \( \frac{1}{v(t)^3} = o \left( \frac{\kappa(t)}{t} \right), \ t \to +\infty, \) using (15) and (12) we have for \( |z| = \rho: \)

\[ \rho^q D^q f(z) = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} \left( f(z) + o \left( \frac{\kappa(\nu)}{\nu} M(r, f) \right) \right) \]

\[ + O \left( \frac{\kappa(\nu)}{\nu} M^*(r) \rho^{\nu_1} \right) \]

\[ = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} \left( f(z) + o \left( \frac{\kappa(\nu)}{\nu} M(r, f) \right) \right) \]

\[ + O \left( \frac{\kappa(\nu)}{\nu} M(r, f) \left( \frac{\rho}{r} \right)^{\nu_1} \right) \]  

(16)

as \( r \to +\infty \) outside a set of finite logarithmic measure.

Next we choose \( z_0 \) so that \( |f(z_0)| = M(r, f) \) and take \( \eta = 1, \tau = \ln(\rho/r). \) Then Theorem 3 gives

\[ \ln \left| f \left( \frac{\rho}{r} z_0 \right) \right| = \ln |f(z_0)| + \nu \tau + O(1), \ |\tau| \leq \frac{1}{18\kappa(\nu)}, \]

so that

\[ \ln M(\rho, f) \geq \ln M(r, f) + \nu \ln(\rho/r) + O(1). \]

Since \( (\rho/r)^{\nu_1 - \nu} = \exp\{\tau(\nu_1 - \nu)\} = O(1), \) we have

\[ \left( \frac{\rho}{r} \right)^{\nu_1} M(r, f) = \left( \frac{\rho}{r} \right)^{\nu} \left( \frac{\rho}{r} \right)^{\nu_1 - \nu} M(r, f) \]

\[ = O \left( \left( \frac{\rho}{r} \right)^{\nu} M(r, f) \right) = O(M(\rho, f)). \]

Thus, (16) yields

\[ \rho^q D^q f(z) = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} \left( f(z) + O \left( \frac{\kappa(\nu)}{\nu} M(\rho, f) \right) \right) \]  

(17)

According to (11) we have

\[ \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - q)} = (1 + o(1)) \nu^q \left( 1 + O \left( \frac{1}{\nu} \right) \right), \ \nu \to +\infty \]  

(18)
Hence
\[
\rho^q D^q f(z) = \nu^q \left( 1 + O\left( \frac{1}{\nu} \right) \right) \left( f(z) + O\left( \frac{\kappa(\nu)}{\nu} M(\rho, f) \right) \right)
\]
\[
= \nu^q \left( f(z) + O\left( \frac{\kappa(\nu)}{\nu} M(\rho, f) \right) \right)
\]
when \( r \to +\infty \) outside a set of finite logarithmic measure, that is (4).

We choose \( z \) in (4) in turn so as to make \( |f(z)| \) and \( |D^q f(z)| \) maximal and deduce that
\[
M(\rho, D^q f) \leq \left( 1 + O\left( \frac{\kappa(\nu)}{\nu} \right) \right) \left( \nu^q \right) M(\rho, f)
\]
and
\[
M(\rho, D^q f) \geq \left( 1 + O\left( \frac{\kappa(\nu)}{\nu} \right) \right) \left( \frac{\nu}{\rho} \right)^q M(\rho, f)
\]
so that
\[
M(\rho, D^q f) = \left( 1 + O\left( \frac{\kappa(\nu)}{\nu} \right) \right) \left( \frac{\nu}{\rho} \right)^q M(\rho, f).
\]

To complete the proof of (5) it remains to show that
\[
\ln M(\rho, f) = \ln M(r, f) + \nu \ln(\rho/r) + o(1).
\]
To see this we note that (7) and (12) yield for our range of \( \rho \)
\[
\ln M(\rho, f) = \nu_1 \ln \rho + \ln M(\rho, P) + o(1).
\]
On the other hand it follows from Lemma 2 that
\[
M(\rho, P) = M(r, P) \left( 1 + O\left( \frac{(\rho - r)\kappa(\nu)}{r} \right) \right) \sim M(r, P)
\]
if \( \kappa(\nu) \ln(\rho/r) = o(1) \), and now the second equality of (5) also follows and the proof of Theorem 2 is complete.

**Remark 1.** \( D^q(\rho^q f(z)) \) has the same asymptotic estimate as \( \rho^q D^q f(z) \), thus under the conditions of Theorem 2 for \( |z| = \rho \) we have
\[
D^q(\rho^q f(z)) = \nu^q \left( f(z) + O\left( \frac{\kappa(\nu)}{\nu} M(\rho, f) \right) \right),
\]
as \( r \to +\infty \) outside a set of finite logarithmic measure. Note that the operator \( D^q(\rho^q f(\rho e^{i\varphi})) \) keeps analyticity and have other nice properties (see [3, Ch. IX]).
3. An Application to Fractional Differential Equations

It is known [16, 6, 2] that every nontrivial solution of the equation

$$f^{(q)}(z) + a(z)f(z) = 0,$$

where $a(z)$ is a polynomial of degree $m$, is an entire function of order $\rho[f] = 1 + \frac{m}{q}$, where

$$\rho[f] = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

On the other hand, for fractional values of $q \in (0, 1)$ equation (20) with $a(t) = A(t^q)$, where $A$ is a polynomial of degree $m$, admits a solution of the form $f(t) = v(t^q)$, $t \geq 0$, where $v$ is entire with $\rho[v] \leq \frac{1+m}{q}$ ([5]).

It is not possible to estimate the growth of solutions of (20) using Theorem 2, because it would require an asymptotics for the Gelfond-Leontiev differential operators (see [5]), which is more general than $D^q$. Nevertheless we can obtain an asymptotic of solutions for some class of fractional equations.

We consider the fractional differential equation in the form

$$\tilde{D}^q(r^q f(z)) + a(z)f(z) = 0,$$

where the coefficient $a(z)$ is an entire function, $q > 0$, and

$$\tilde{D}^q f(z) = D^q f(z) - \Gamma(q + 1)f(0).$$

**Remark 2.** The analog of the operator (22) can be found in ([3, Chap.9]). This definition provides that $\tilde{D}(r^q f(re^{i\varphi}))\big|_{r=0} = 0$.

The proofs of the following theorems are standard ([6]).

**Theorem 4.** The equation (21) with the initial condition $f(0) = f_0$ has an entire solution.

**Theorem 5.** Let $a(z)$ be a polynomial of degree $m \geq 0$. Then all nontrivial solutions $f$ of the equation (21) have the order of growth $\varrho = \frac{m+1}{q}$. 
References


