International Journal of Applied Mathematics

Volume 29 No. 3 2016, 381-387

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v29i3.9

EXPONENTIAL STABILITY TO A QUASILINEAR HYPERBOLIC EQUATION WITH INTERNAL DAMPING

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Abstract: In this work we prove that the effect of the memory together with the frictional damping produces stabilization for the system of two identical laminated beams of uniform density taking into account that an adhesive of small thickness is bonding the two beams and produce the interfacial slip. It is assumed that the thickness of the adhesive is small enough so that the contribution of its mass to the kinetic energy of the entire beam may be ignored.

AMS Subject Classification: 35B40, 35L72, 35B35

Key Words: exponential stability, quasilinear hyperbolic equation, energy method

1. Introduction

We consider the initial boundary value problem associated to the damped quasi-

Received: April 20, 2016

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linear hyperbolic equation

$$u_{tt} - \Delta u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u_{tt} + u_t = 0, \tag{1}$$

in the cylinder $Q = \Omega \times]0, T[$, $T > 0$ is a real number, Ω is a smooth bounded domain in $\mathbf{R}^n$, M is a real valued function with $M(\lambda) > \rho > 0$, $\forall \lambda \geq 0$ and some $\rho > 0$. The equation (1) without the dissipative term u_t and $M(\lambda) \equiv 1$ arises in the study of the extensional vibrations of thin rods, see [9]. For the equation with $M(\lambda) = \lambda_0$, with $\lambda_0 = \int_\Omega \phi^2(x)dx$, where ϕ is the torsion function arises of the torsional vibrations of thin rods, see [5]. The function $M(\lambda)$ in (1) has its motivation in mathematical description of the vibrations of an elastic stretched string, that is, the equation

$$u_{tt} - \Delta u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = 0,$$

what for $(M(\lambda) \geq \rho > 0)$ and was studied [10, 7, 4]. The situation $(M(\lambda) \geq 0)$ was treated by [1, 2, 6, 3], among others. The equation

$$u_{tt} - \Delta u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u_{tt} = f,$$

with $M(\lambda) \geq \rho > 0$ was studied in [8] where the existence and uniqueness of global classical solutions were provided. In this work we prove that the global classical solutions of equation (1) decay exponentially using the same technique as in [11].

2. Existence of Solution

Let $(w_j)_{j\in\mathbf{N}}$ be a system of eigenfunctions of $-\Delta$ which is defined on $H_0^1(\Omega) \cap H^2(\Omega)$. We denote by $V \equiv V(\Omega)$ the set of all finite combinations of $(w_j)_{j\in\mathbf{N}}$. Putting $(u, v) = \int_\Omega u(x)v(x)dx$ and $(u, v)_m = ((-\Delta)^m u, v), m = 1, 2, ...$, then $(\cdot,\cdot)_m$ is a inner product on V. We put $V_m \equiv V_m(\Omega)$ as the closure of V by the topology of norm $|\cdot|_m^2 = (\cdot,\cdot)_m$. Then we have that

$$H_0^1(\Omega) \equiv V_1 \supset V_2 \supset ... \supset V_m \supset ...,$$

$V_m \subset H^m(\Omega), m = 1, 2, ...$, and the norm $|\cdot|_m$ is equivalent in V_m to the standard norm of $H^m(\Omega)$ and all the above injections are compact.

Consider the following hypotheses about the real valued function M:

$(H.1)$ $M \in C^1[0,\infty)[$, and there exist constants $\alpha > 0$ and $\rho > 0$ that verify $M(\lambda) \geq \alpha\lambda^{1/2} + \rho$, $\forall\lambda \in [0,\infty[$,

$(H.2)$ $|M'(\lambda)|\lambda^{1/2} \leq \beta(\lambda)M(\lambda)$ where $\beta \in C^0[0,\infty[$, $\beta(\lambda) \geq 0$, $\lambda \geq 0$.

Theorem 1. *Under the hypotheses $(H.1), (H.2)$ there exists a unique solution of the initial boundary value problem associated to the equation (1), with initial data $u_0, u_1 \in V_m, m \geq 2$, in the class $u \in C^2(0,T;V_m)$, that verifies*

$$u_{tt} - \Delta u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u_{tt} + u_t = 0 \quad in \quad Q.$$

Proof. The proof can be made by the Faedo-Galerkin method and compactness argument with same idea as in [8]. □

3. Asymptotic Behavior

In this section we use the same technique as in [11] to prove the exponential stability of the solution. In the sequel we have two lemmas.

Lemma 2. *The energy $E(t) = |u_t|^2 + \|u\|^2$ associated the equation (1) satisfies*

$$E'(t) \leq -|u_t|^2 + \alpha^2 C_1^2\|u\|^2, \tag{2}$$

where C_1 is a positive constant.

Proof. Taking the derivative of the energy $E(t)$ with respect to t we obtain:

$$\begin{aligned} E'(t) &= 2(u_{tt}, u_t) + 2((u, u_t)) = 2(u_{tt}, u_t) - 2(\Delta u, u_t) \\ &= 2(u_{tt} - \Delta u, u_t) = 2(M(\|u\|^2)\Delta u_{tt} - u_t, u_t) \\ &= -2|u_t|^2 - 2M(\|u\|^2)(-\Delta u_{tt}, u_t). \end{aligned}$$

By (H.1) we have $M(\|u\|^2) \geq \alpha\|u\|^2$, then $-M(\|u\|^2) \leq -\alpha\|u\|^2$. Thus,

$$\begin{aligned} E'(t) &\leq -2|u_t|^2 - 2\alpha\|u\|(-\Delta u_{tt}, u_t) \\ &\leq -2|u_t|^2 + 2\alpha\|u\||\Delta u_{tt}||u_t| \\ &\leq -2|u_t|^2 + \alpha^2 C_1^2\|u\|^2 + |u_t|^2. \end{aligned}$$

That is,

$$E'(t) \leq -|u_t|^2 + \alpha^2 C_1^2 \|u\|^2,$$

where $C_1 > 0$ is a constant such that $|\Delta u_{tt}| \leq C_1$. □

Lemma 3. *Consider $G(t) = (u_t, u)$, then*

$$G'(t) \leq \frac{3}{2}|u_t|^2 + (\alpha C_0 C_1 + \frac{C_0^2}{2} - 1)\|u\|^2, \tag{3}$$

where C_0 is a constant from Poincarè's inequality.

Proof. Taking the derivative of $G(t)$ with respect to t we obtain:

$$\begin{aligned} G'(t) &= (u_{tt}, u) + |u_t|^2 \\ &= (\Delta u + M(\|u\|^2)\Delta u_{tt} - u_t, u) + |u_t|^2 \\ &= \|u\|^2 - M(\|u\|^2)(-\Delta u_{tt}, u) - (u_t, u) + |u_t|^2. \end{aligned}$$

By (H.1), we have

$$-M(\|u\|^2)(-\Delta u_{tt}, u) \leq \alpha C_0 C_1 \|u\|^2,$$

where C_0 is a constant of Poincarè's inequality.

Now

$$(u_t, u) \leq \frac{1}{2}|u_t|^2 + \frac{1}{2}|u|^2 \leq \frac{1}{2}|u_t|^2 + \frac{C_0^2}{2}\|u\|^2.$$

Hence

$$G'(t) \leq \frac{3}{2}|u_t|^2 + (\alpha C_0 C_1 + \frac{C_0^2}{2} - 1)\|u\|^2.$$

□

Now we are in position of to prove our principal result.

Theorem 4. *Let $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, then for each $\varepsilon > 0$, small enough, there are constants $\gamma(\varepsilon) > 0$ and $C(\varepsilon) > 0$ such that the solution of the equation (1) satisfies*

$$|u_t|^2 + \|u\|^2 \leq C(\varepsilon)e^{-\gamma(s)t}, \quad \forall t \geq 0.$$

Proof. Consider a small perturbation of the energy functional, given by

$$E_\varepsilon(t) = E(t) + \varepsilon G(t).$$

For $\varepsilon > 0$ small enough, that is

$$\varepsilon < \min\left\{\frac{2}{3}, \frac{2\alpha^2 C_1^2}{2 - 2\alpha C_0 C_1 - C_0^2}, \frac{2}{C_0^2}\right\},$$

we have

$$\begin{aligned} E_\varepsilon(t) &= |u_t|^2 + \|u\|^2 + \varepsilon(u_t, u) \\ &\leq |u_t|^2 + \|u\|^2 + \frac{\varepsilon}{2}|u_t|^2 + \frac{\varepsilon C_0^2}{2}\|u\|^2. \end{aligned}$$

So,

$$E_\varepsilon(t) \leq (1 + \frac{\varepsilon}{2})|u_t|^2 + (1 + \frac{\varepsilon C_0^2}{2})\|u\|^2.$$

Thus,

$$E_\varepsilon(t) \leq a(\varepsilon)E(t), \tag{4}$$

where

$$a(\varepsilon) = \max\left\{1 + \frac{\varepsilon}{2}, 1 + \frac{\varepsilon C_0^2}{2}\right\} > 0.$$

Now,

$$\begin{aligned} E'_\varepsilon(t) &= E'(t) + \varepsilon G'(t) \\ &\leq -|u_t|^2 + \alpha^2 C_1^2\|u\|^2 + \frac{3\varepsilon}{2}|u_t|^2 + \varepsilon\left(\alpha C_0 C_1 + \frac{C_0^2}{2} - 1\right)\|u\|^2 \\ &= (\frac{3\varepsilon}{2} - 1)|u|^2 + \left[\alpha^2 C_1^2 + \varepsilon\left(\alpha C_0 C_1 + \frac{C_0^2}{2} - 1\right)\right]\|u\|^2. \end{aligned}$$

Then,

$$E'_\varepsilon(t) \leq b(\varepsilon)E(t), \tag{5}$$

where

$$b(\varepsilon) = \max\left\{\frac{3\varepsilon}{2} - 1, \alpha^2 C_1^2 + \varepsilon\left(\alpha C_0 C_1 + \frac{C_0^2}{2} - 1\right)\right\}$$

and $b(\varepsilon) < 0$ by the hypothesis about ε.

Multiplying (4) and (5) by $-b(\varepsilon)$ and $a(\varepsilon)$ respectively, and summing, we obtain

$$a(\varepsilon)E'_\varepsilon(t) - b(\varepsilon)E_\varepsilon(t) \leq 0.$$

Hence,

$$E_\varepsilon(t) \leq E_\varepsilon(0)e^{-\gamma(s)t}, \quad \forall t \geq 0, \tag{6}$$

where $\gamma(\varepsilon) = -\dfrac{b(\varepsilon)}{a(\varepsilon)} > 0$.

Now,

$$\begin{aligned} E_\varepsilon(t) &= |u_t|^2 + \|u\|^2 + \varepsilon(u_t, u) \\ &\geq (1 - \frac{\varepsilon}{2})|u_t|^2 + (1 - \frac{\varepsilon C_0^2}{2})\|u\|^2 \\ &\geq \beta(\varepsilon)E(t), \end{aligned}$$

where $\beta(\varepsilon) = \min\left\{1 - \dfrac{\varepsilon}{2}, 1 - \dfrac{\varepsilon C_0^2}{2}\right\}$.

Thus, by (6) we have

$$E(t) \leq \beta^{-1}(\varepsilon)E_\varepsilon(t) \leq \beta^{-1}(\varepsilon)E_\varepsilon(0)e^{-\gamma(s)t}, \quad \forall t \geq 0.$$

Therefore,

$$E(t) \leq C(\varepsilon)e^{-\gamma(s)t}, \quad \forall t \geq 0,$$

with $C(\varepsilon) = \beta^{-1}(\varepsilon)E_s(0)$, that is

$$|u_t|^2 + \|u\|^2 \leq C(\varepsilon)e^{-\gamma(s)t}, \quad \forall t \geq 0.$$

□

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