ON THE CONVERGENCE OF
DIRICHLET SERIES WITH RANDOM EXPONENTS

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Abstract: For the Dirichlet series of the form $F(z, \omega) = \sum_{k=0}^{+\infty} f_k(\omega)e^{z\lambda_k(\omega)}$ $(z \in \mathbb{C}, \omega \in \Omega)$ with pairwise independent real exponents $(\lambda_k(\omega))$ on probability space $(\Omega, \mathcal{A}, P)$ an estimates of abscissas convergence and absolutely convergence are established.

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1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\Lambda = (\lambda_k(\omega))_{k=0}^{+\infty}$ and $f = (f_k(\omega))_{k=0}^{+\infty}$ sequences of positive and complex-valued random variables on it, respectively. Let $\mathcal{D}$ be the class of formal random Dirichlet series of the form

$$f(z) = f(z, \omega) = \sum_{k=0}^{+\infty} f_k(\omega)e^{z\lambda_k(\omega)}$$ $(z \in \mathbb{C}, \omega \in \Omega)$.

Let $\sigma_c(f, \omega)$ and $\sigma(f, \omega)$ be the abscissa of convergence and absolute convergence of this series for fixed $\omega \in \Omega$, respectively. The simple modification of [1]–[3] one has that for the Dirichlet series $f \in \mathcal{D}$ for fixed $\omega \in \Omega$ such that

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\[ \lambda_k(\omega) \to +\infty \ (k \to +\infty) \]

\[ \sigma(f, \omega) \leq \sigma_c(f, \omega) \leq \alpha_0(\omega) := \lim_{k \to +\infty} \frac{-\ln |f_k(\omega)|}{\lambda_k(\omega)} \]

\[ \leq \sigma(f, \omega) + \tau(\omega, \Lambda), \quad (1) \]

or in the case \(-\ln |f_k(\omega)| \to +\infty \ (k \to +\infty)\)

\[ (1-h)\sigma_c(f, \omega) \leq (1-h)\alpha_0(\omega) \leq \sigma(f, \omega), \quad h = h(\omega, f), \quad (2) \]

where \(\tau(\omega, \Lambda) := \lim_{k \to \infty} \frac{\ln k}{\lambda_0(\omega)}\), \(h(\omega, f) := \lim_{k \to \infty} -\frac{\ln k}{\ln |f_k(\omega)|}\).

Also,

\[ \sigma_c(f, \omega) = \sigma(f, \omega) = \alpha_0(\omega) \quad (3) \]

for fixed \(\omega \in \Omega\) such that \(\tau(\omega) = 0\) or \(\ln k/(-\ln |f_k(\omega)|) \to +0 \ (k \to +\infty)\). Remark, that from condition \(\tau(\omega) < +\infty\) we get \(\lambda_k(\omega) \to +\infty \ (k \to +\infty)\). In the case \(\sigma_c(f, \omega) > 0\) the series of the form \(\sum_{k=0}^{+\infty} f_k(\omega)\) is convergent, thus \(-\ln |f_k(\omega)| \to +\infty \ (k \to +\infty)\).

The following assertion is proved in [3, Corollary 5] (another version [2, Theorem 1]) in the case of the deterministic Dirichlet series with sequence of exponents that increase to infinity, i.e., \(f_k(\omega) \equiv f_k \in \mathbb{C} \ (k \geq 0)\) and \(\lambda_k(\omega) \equiv \lambda_k\), \(0 \leq \lambda_k < \lambda_{k+1} \to +\infty \ (0 \leq k \to +\infty)\).

**Proposition 1.** Let \(f \in D\). Then \(\sigma_a(f, \omega) \leq \sigma_c(f, \omega) \leq \alpha_0(\omega) \ (\forall \omega \in \Omega)\), and

\[ \sigma_a(f, \omega) \geq \gamma(\omega)\alpha_0(\omega) - \delta(\omega) \geq \gamma(\omega)\sigma_c(f, \omega) - \delta(\omega) \quad (4) \]

for arbitrary real random variables \(\gamma, \delta\) and for all \(\omega \in \Omega\) such that \(\gamma(\omega) > 0\) and

\[ \sum_{k=0}^{+\infty} |f_k(\omega)|^{1-\gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)} < +\infty. \quad (5) \]

**Remark 2.** Condition (5) implies, that \((\gamma(\omega)-1)\ln |f_k(\omega)| + \delta(\omega)\lambda_k(\omega) \to +\infty \ (k \to +\infty)\) for such \(\omega\). But, in general, from this condition don’t follows neither \(\lambda_k(\omega) \to +\infty\) nor \(\ln |f_k(\omega)| \to +\infty \ (k \to +\infty)\).

**Proof of Proposition 1.** It is obvious that \(\sigma(f, \omega) \leq \sigma_c(f, \omega)\).

We prove now that \(\sigma_c(f, \omega) \leq \alpha_0(\omega)\). Indeed, assume first that \(\alpha_0(\omega) \neq +\infty\) and put \(x_0 = \alpha_0(\omega) + \varepsilon\), where \(\varepsilon > 0\) is arbitrary. Then, \(|f_k(\omega)|e^{x_0\lambda_k(\omega)} = \exp\{\lambda_k(\omega)(\ln |f_k(\omega)|/\lambda_k(\omega) + x_0)\}\). But by definition of \(\alpha_0(\omega)\) there exists a
By definition of $\alpha_k \ln k$ and for some sequence $\lambda(\sigma)$ there
hyperbolic constant $c > 0$ such that
$$\ln |f_k(\omega)|/\lambda_k(\omega) > -(\alpha_0(\omega) + \varepsilon/2)$$
(k = $k_j$, $j \geq 1$). Thus,
$$|f_k(\omega)|/\lambda_k(\omega) + x_0 > \varepsilon/2 \quad (k = k_j, j \geq 1),$$
and
$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} \geq e^{\lambda_k(\omega)\varepsilon/2} \geq 1 \quad (k = k_j, j \geq 1),$$
therefore $\sigma_c(f, \omega) \leq \alpha_0(\omega) + \varepsilon$, but $\varepsilon > 0$ is arbitrary.

The case $\alpha_0(\omega) = +\infty$ is trivial. In the case $\alpha_0(\omega) = -\infty$ for every $E > 0$
and for some sequence $k_j \to +\infty$ ($j \to +\infty$) by definition $\alpha_0(\omega)$ we obtain
$$\ln |f_k(\omega)|/\lambda_k > E \quad (k = k_j, j \geq 1).$$
Therefore $|f_k(\omega)|\exp\{E\lambda_k\} > 1 \quad (k = k_j, j \geq 1)$, i.e. the
Dirichlet series diverges at the point $z = -E$, but $E > 0$ is arbitrary. Thus, $\sigma_c = -\infty$.

Let now $x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega)$ for arbitrary $\varepsilon > 0$. Then,
$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} = |f_k(\omega)|^{1 - \gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}\left(|f_k(\omega)|e^{(\alpha_0(\omega) - \varepsilon)\lambda_k(\omega)}\right)^{\gamma(\omega)}. \quad (6)$$
By definition of $\alpha_0(\omega)$, we obtain $\alpha_0(\omega) < -\frac{\ln f_k(\omega)}{\lambda_k(\omega)} + \varepsilon/2$ for $k \geq k_0(\omega)$, and
thus $|f_k(\omega)|e^{(\alpha_0(\omega) - \varepsilon)\lambda_k(\omega)} < \exp\{-\lambda_k\varepsilon/2\} \leq 1 \quad (k \geq k_0(\omega))$. Hence by (6) one
has $|f_k(\omega)|e^{x_0\lambda_k(\omega)} \leq |f_k(\omega)|^{1 - \gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}$ and by condition (5) we obtain
$$\sigma(f, \omega) \geq x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega).$$
But, $\varepsilon > 0$ is arbitrary.

From Proposition 1 it simply follows such a statement.

**Proposition 3.** Let $f \in \mathcal{D}$. Then equalities (3) hold for all $\omega \in \Omega$ such, that
$$\ln k = o(\ln |f_k(\omega)|) \quad (k \to +\infty). \quad (7)$$

**Remark 4.** If the sequences $\Lambda$ and $\mathbf{f}$ such that $(|f_k(\omega)|e^{x_0\lambda_k(\omega)})$
are the sequences of independent random variables for every $x \in \mathbb{R}$, then by Kolmogorov’s
Zero-One Law ([4]) random variable $\sigma(f, \omega)$ is almost surely (a.s.) constant.
That is, $\sigma(f, \omega) = \sigma \in \{-\infty, +\infty\}$ a.s. In the book [4] it is written when $\Lambda$
monotonic increasing to infinity sequence $\lambda_k(\omega) \equiv \lambda_k$. The same we get when
$$-\frac{\ln |f_k(\omega)|}{\lambda_k(\omega)}$$
is the sequence of independent random variables, and $\tau(\omega, \Lambda) = 0$
or $h(\omega, \mathbf{f}) = 0$. It follows from Proposition 3 and equalities (3).

In the papers [5]–[10] considered question about abscissas of convergence
random Dirichlet series from the class $\mathcal{D}$ in case, when $\Lambda_+ = (\lambda_k)$ is increasing
sequence of positive numbers, i.e., $0 = \lambda_0 < \lambda_k < \lambda_{k+1} \to +\infty \quad (1 \leq k \to +\infty)$
and $\tau(\omega, \Lambda) \equiv \tau(\Lambda) < +\infty$.

We have such elementary assertion.
Proposition 5. Let \( f \in D(\Lambda) \) be a Dirichlet series of the form
\[
f(z, \omega) = \sum_{k=0}^{\infty} a_k Z_k(\omega) e^{z \lambda_k(\omega)},
\]
where \( (Z_k(\omega)) \) is a sequence of random complex-valued variables.

1. If the condition \( \tau(\omega, \Lambda) = 0 \) holds and
\[
\lim_{k \to +\infty} -\frac{\ln |Z_k(\omega)|}{\lambda_k(\omega)} = 0 \quad \text{a.s.,}
\]
then \( \sigma_c(f, \omega) = \sigma(f, \omega) = \alpha_0^* := \lim_{k \to +\infty} -\frac{\ln |a_k|}{\lambda_k(\omega)} \quad \text{a.s.} \)

2. If \( \alpha_0(\omega) = +\infty \) and the conditions \( \tau(\omega, \Lambda) < +\infty \),
\[
\lim_{k \to +\infty} -\frac{\ln |Z_k(\omega)|}{\lambda_k(\omega)} > -\infty \quad \text{a.s.}
\]
hold, then \( \sigma(f, \omega) = +\infty \) a.s.

We obtain Proposition 5 immediately from inequalities (1).

In the paper [6], it is considered only 1 for the case of the Dirichlet series \( f \in D(\Lambda_+) \) of the form
\[
f(z) = \sum_{k=0}^{+\infty} a_k Z_k(\omega) e^{z \lambda_k}.
\]

From Proposition 5, in particular, they follow Theorem 1 (when \( \alpha_0 := \alpha_0^* = +\infty \)) and Theorem 3 (when \( \alpha_0 = 0 \)) from [6], which are proved under such condition for expectation:
\[
\begin{align*}
(\exists \alpha > 0, \beta > 0) & : \sup\{ E|Z_k|^{\alpha}, E|Z_k|^{-\beta} : k \geq 0\} < +\infty. \\
\end{align*}
\]

By the Bienayme-Chebyshev inequality ([11, 12]) and the Borel-Cantelli Lemma ([4], also about refined Second Borel-Cantelli lemma see [13]) from condition (10) it easy follows, that a.s. for all enough large \( k \) inequalities \( k^{-\gamma} \leq |Z_k(\omega)| < k^{\gamma} \) with \( \gamma = \max\{2/\alpha, 2/\beta\} \) hold, and if \( \tau(\Lambda) = 0 \), then and condition (8). Similarly, if \( \tau(\Lambda) < +\infty \), then from condition (\( \exists \beta > 0 \)):
\[
\begin{align*}
\sup\{ E|Z_k|^{-\beta} : k \geq 0\} < +\infty & \text{ follows condition (9).}
\end{align*}
\]

It should be noted, that condition (8) follows from such condition (see [10]) on sequence of distribution functions of random variables \( (|Z_k(\omega)|) \),
\[
\begin{align*}
(\forall \varepsilon > 0) & : \sum_{k=0}^{+\infty} (1 - F_k^*(e^{\varepsilon \lambda_k}) + F_k^*(e^{-\varepsilon \lambda_k})) < +\infty,
\end{align*}
\]
where \( F_k^*(x) := P\{\omega : |Z_k(\omega)| < x\} \). In particular, from this condition it follows
\[
\lim_{k \to +\infty} F_k^*(+0) = 0.
\]

In the papers [7]–[9] in the case of independent random variables \( f = (f_k) \), besides, generalized on class \( D(\Lambda) \) assertion of known Blackwell’s conjecture on power series with random coefficients, proved in [14] (see also [4]).
In the general case, for Dirichlet series from the class $\mathcal{D}(\Lambda_+)$ in [10] (see also similar results for random gap power series in [15]–[18]) two theorems are proved. In particular, we find ([10]) the following theorem.

**Theorem 6 ([10]).** Let $f \in \mathcal{D}(\Lambda_+)$ and $f = (f_k(\omega))$ be a sequence such that $(|f_k(\omega)|)$ is the sequence of pairwise independent random variables with functions of distribution $F_k(x) := P\{\omega : |f_k(\omega)| < x\}$, $x \in \mathbb{R}$, $k \geq 0$. The following assertions are true:

a) If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, +\infty)$ a.s., then
   
   \[ \sum_{k=0}^{+\infty} (1 - F_k((e^{-\rho} + \varepsilon)^{\lambda_k})) < \infty. \]

b) If there exists a sequence $(\delta_k)$: $\delta_k > -\infty$ ($k \geq 0$), $\lim_{k \to +\infty} \delta_k = e^{-\rho}$, $\rho \in (-\infty, +\infty]$, and $\sum_{k=0}^{+\infty} (1 - F_k(\delta_k^{\lambda_k})) = +\infty$, then $\sigma(f, \omega) \leq \rho$ a.s.

Another theorem in [10] contains the converse statements.

In this paper we prove similar theorems for Dirichlet series with random exponents $(\lambda_k(\omega))$ and deterministic coefficients $f = (f_k)$, $f_k \in \mathbb{C}$, $k \geq 0$. Note that in paper [19] a power series of the form $\sum_{k=0}^{+\infty} z^{X_k(\omega)}$ is studied, where $(X_k(\omega))$ is a strictly increasing integer-valued stochastic process.

### 2. The Main Results: Series with Random Exponents

In this section we assume that $f_k(\omega) \equiv f_k \in \mathbb{C}$ ($k \geq 0$) and condition $\ln k = o(\ln |f_k|)$ ($k \to +\infty$) holds, that condition (7) is satisfied for all $\omega \in \Omega$, therefore by Proposition 3 equalities (3) for every $\omega \in \Omega$ hold.

**Theorem 7.** Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x) := P\{\omega : \lambda_k(\omega) < x\}$, $x \in \mathbb{R}$, $k \geq 0$. The following assertions hold:

i) If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s. then
   
   \[ \sum_{k=0}^{+\infty} (1 - F_k((\ln |f_k|/(\rho - \varepsilon))) < \infty. \]

ii) If $0 \geq \sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, 0]$ a.s. then
   
   \[ \sum_{k=0}^{+\infty} F_k((\ln |f_k|/(\rho - \varepsilon))) < \infty. \]

**Proof of Theorem 7.** i) If $\sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s., then from (3) we have $(\exists B \in \mathcal{A}, P(B) = 1)(\forall \omega \in B)$: $\lim_{k \to +\infty} - \ln |f_k|/\lambda_k(\omega) \geq \rho$, and by definition of
\[ \lim, \]
\[ (\forall \omega \in B)(\forall \varepsilon \in (0, \rho))(\exists k^{*}(\omega) \in \mathbb{N})(\forall k \geq k^{*}(\omega)): \]
\[ \lambda_{k}(\omega) < \ln |f_{k}|/(-\rho + \varepsilon). \]  
(11)

We denote
\[ A_{k} := \left\{ \omega: \lambda_{k}(\omega) \geq \frac{\ln |f_{k}|}{(-\rho + \varepsilon)} \right\}. \]

It is clear, that \( B \subset \overline{C} := \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_{k} \), hence \( P(\overline{C}) = 1 \), and \( C = \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_{k} \) is the event “\((A_{k})\) infinitely often”, i.e. \( \overline{C} \) is the event “\((A_{k})\) finitely often”. From pairwise independence of random variables \((\lambda_{k}(\omega))\) follows pairwise independence of events \((A_{k})\). Therefore, by refined Second Borel-Cantelli Lemma ([13, p.84])
\[ \sum_{k=0}^{+\infty} (1 - F_{k}(\ln |f_{k}|/(-\rho + \varepsilon))) = \sum_{k=0}^{+\infty} P(A_{k}) < +\infty. \]

ii) If \( 0 \geq \sigma(\omega, f) \geq \rho \in (-\infty, 0] \) a.s., then instead of (11) we obtain
\[ (\exists B, P(B) = 1)(\forall \omega \in B)(\forall \varepsilon > 0)(\exists k^{*}(\omega) \in \mathbb{N})(\forall k \geq k^{*}(\omega)): \]
\[ \lambda_{k}(\omega) > \ln |f_{k}|/(-\rho + \varepsilon). \]

Therefore, for \( A_{k} := \left\{ \omega: \lambda_{k}(\omega) \leq \ln |f_{k}|/(-\rho + \varepsilon) \right\} \) by the refined Second Borel-Cantelli lemma we obtain again
\[ \sum_{k=0}^{+\infty} F_{k}(\ln |f_{k}|/(-\rho + \varepsilon)) = \sum_{k=0}^{+\infty} P(A_{k}) < +\infty. \]

This completes the proof of Theorem 7. \( \square \)

**Remark 8.** If \( \sigma(f, \omega) > \rho \in [0, +\infty) \) a.s., then from (3) by definition of \( \lim \) we have \((\forall \omega \in B)(\exists \varepsilon^{*} = \varepsilon^{*}(\omega) > 0)(\exists k^{*}(\omega) \in \mathbb{N})(\forall k \geq k^{*}(\omega)): \lambda_{k}(\omega) < \ln |f_{k}|/(-\rho + \varepsilon^{*})\), and similarly as in proof of i) we obtain
\[ \sum_{k=0}^{+\infty} (1 - F_{k}(-\ln |f_{k}|/\rho)) < +\infty \]
in the case \( \rho > 0 \) and in the case \( \rho = 0 \) one has
\[ \sum_{k=0}^{+\infty} (1 - F_{k}(+0)) < +\infty, \]
i.e., in particular, \( \lim_{k \to +\infty} F_{k}(+0) = 1 \). Namely, if \( \lim_{k \to +\infty} F_{k}(+0) < 1 \), then \( \sigma(f, \omega) \leq 0 \) a.s.
Theorem 9. Let $\Lambda = (\lambda_k(\omega))$ be a sequence of random variables with distribution functions $F_k(x) := P\{ \omega : \lambda_k(\omega) < x \}$, $x \in \mathbb{R}$, $k \geq 0$, and $f \in \mathcal{D}(\Lambda)$. The following assertions hold:

i) If there exist $\rho \in (0, +\infty)$ and a sequence $(\varepsilon_k)$ such that $\varepsilon_k \rightarrow +0 \ (k \rightarrow +\infty)$ and $\sum_{k=0}^{+\infty} (1 - F_k(\frac{|f_k|}{\varepsilon_k})) < +\infty$, then $\sigma(f, \omega) \geq \rho$ a.s.

ii) If there exist $\rho \in (-\infty, 0]$ and a sequence $(\varepsilon_k)$ such that $\varepsilon_k \rightarrow +0 \ (k \rightarrow +\infty)$ and $\sum_{k=0}^{+\infty} F_k(\frac{|f_k|}{\rho + \varepsilon_k}) < +\infty$, then $\sigma(f, \omega) \geq \rho$ a.s.

Proof of Theorem 9. i) We note $1 - F_k(\frac{|f_k|}{(-\rho + \varepsilon_k)}) = P(A_k)$, where

$A_k := \{ \omega : \lambda_k(\omega) \geq \ln |f_k|/(-\rho + \varepsilon_k) \}$.

Therefore, from condition one has $\sum_{k=0}^{+\infty} P(A_k) < \infty$. Thus, by the first part of Borel-Cantelli Lemma $P(\overline{\mathcal{C}}) = 1$, $\mathcal{C} := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k$. Since, $\overline{\mathcal{C}} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A_k}$, then for all $\omega \in \mathcal{C}$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A_k}$ and $-\rho + \varepsilon_k < 0$ for all $k \geq k^*(\omega)$. Here, $(\forall k \geq k^*(\omega)) : \lambda_k(\omega) < \frac{\ln |f_k|}{-\rho + \varepsilon_k}$. Using

$\frac{-\ln |f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$, we get

$$\sigma(f, \omega) = \lim_{k \rightarrow +\infty} \frac{-\ln |f_k|}{\lambda_k(\omega)} \geq \lim_{k \rightarrow +\infty} (\rho - \varepsilon_k) = \rho \quad \text{a.s.} \tag{12}$$

ii) By the condition $\sum_{k=0}^{+\infty} P(A_k) < +\infty$, where

$A_k := \{ \omega : \lambda_k(\omega) < \ln |f_k|/(-\rho + \varepsilon_k) \}$.

Since, by the first part of Borel-Cantelli Lemma

$$P(\overline{\mathcal{C}}) = 1, \quad \mathcal{C} := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k.$$  

Where, as above for every $\omega \in \overline{\mathcal{C}} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A_k}$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A_k}$ and $-\rho + \varepsilon_k > 0$ for all $k \geq k^*(\omega)$, such hat, $(\forall k \geq k^*(\omega)) : \lambda_k(\omega) \geq \frac{\ln |f_k|}{-\rho + \varepsilon_k}$. Hence, $\frac{-\ln |f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$ and, therefore, we have again the “chain” of relations (12).

The proof of Theorem 9 is complete. \qed
3. Some Corollaries

**Corollary 10.** Let \( f \in \mathcal{D}(\Lambda) \) and \( \Lambda = (\lambda_k(\omega)) \) be a sequence of pairwise independent random variables with distribution functions \( F_k(x) \), \( k \geq 0 \). If \( \lim_{k \to +\infty} F_k(0) < 1 \) and \( f_k \to 0 \) \((k \to +\infty)\), then \( \sigma(f, \omega) = 0 \ a.s. \)

Proof of Corollary 10. By Remark 8, \( \sigma(f, \omega) \leq 0 \ a.s. \) It is remains to prove that \( \sigma(f, \omega) \geq 0 \ a.s. \) Indeed, \( \lambda_k(\omega) \geq 0 \), therefore \( F_k(0) = P\{\omega: \lambda_k(\omega) < 0\} = 0 \). Hence, \( \sum_{k=k_0}^{+\infty} F_k(\ln|f_k|/\varepsilon_k) < +\infty \) because \( \ln|f_k|/\varepsilon_k < 0 \) \((k \geq k_0)\). Thus, by Theorem 9 ii), \( \sigma(f, \omega) \geq 0 \ a.s. \)

Corollary 10 implies immediately the statement of Corollary 11.

**Corollary 11.** Let \( f \in \mathcal{D}(\Lambda) \) and \( \Lambda = (\lambda_k(\omega)) \) be a sequence of pairwise independent random variables with distribution functions \( F_k(x) \), \( k \geq 0 \). If there exists a positive random variable \( a(\omega) \) such that \((\forall x \geq 0)(\forall k \in \mathbb{Z}_+): F_k(x) \leq F_a(x) = P\{\omega: a(\omega) < x\} \) and \( F_a(+0) < 1 \) and \( f_k \to 0 \) \((k \to +\infty)\), then \( \sigma(f, \omega) = 0 \ a.s. \)

**Corollary 12.** Let \( f \in \mathcal{D}(\Lambda) \) and \( \Lambda = (\lambda_k(\omega)) \) be a sequence of random variables with distribution functions \( F_k(x) \), \( k \geq 0 \). If \( f_k \to 0 \) \((k \to +\infty)\) and there exist a positive random variable \( b(\omega) \) and \( \rho > 0 \) such that \((\forall x \geq 0)(\forall k \in \mathbb{Z}_+): F_k(x) \geq F_b(x) = P\{\omega: b(\omega) < x\}, \int_0^{+\infty} n_\mu(t\rho) \ dF_b(t) < +\infty \), where \( n_\mu(t) = \sum_{\mu_k \leq t} 1 \) is the counting function of a sequence \( \mu_k = -\ln|f_k| \), then \( \sigma(f, \omega) \geq \rho \ a.s. \)

Proof of Corollary 12. We remark that

\[
\sum_{k=k_0}^{n} \left(1 - F_k\left(\ln|f_k| / -\rho + \varepsilon_k\right)\right) \leq \int_{\mu_{k_0}}^{\mu_n} \left(1 - F_k\left(t/\rho\right)\right) \ d\mu(t) \\
\leq \int_{\mu_{k_0}}^{\mu_n} \left(1 - F_b(t/\rho)\right) \ d\mu(t) + O(1) \\
= \int_{\mu_{k_0}/\rho}^{\mu_n/\rho} n_\mu(t\rho) \ dF_b(t) + O(1),
\]

\((n \to +\infty)\), because \( -\ln|f_k| > 0 \) \((k \geq k_0)\) and \( \rho - \varepsilon_k < \rho \) for all \( k \geq 0 \). Therefore, the series \( \sum_{k=k_0}^{+\infty} \left(1 - F_k\left(\ln|f_k| / -\rho + \varepsilon_k\right)\right) \) converges. Hence by Theorem 9 ii) we complete the proof. \(\square\)
Corollary 13. Let $\Lambda = (\lambda_k(\omega))$ be a increasing (a.s.) sequence of pairwise independent random variables and $f \in D(\Lambda)$. If $F_0(+0) < 1$, where $F_0$ is distribution function of $\lambda_0(\omega)$, and $f_k \to 0$ ($k \to +\infty$), then $\sigma(f, \omega) = 0$ a.s.

Proof of Corollary 13. We remark that $F_{k+1}(x) \leq F_k(x)$, because $\lambda_k(\omega) \leq \lambda_{k+1}(\omega)$ ($k \geq 0$) a.s. Therefore, by Corollary 11 we obtain the conclusion of Corollary 13. □

References


