SOME RESULTS ON THE $q,k$ AND $p,q$-GENERALIZED GAMMA FUNCTIONS

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Abstract: In this paper, the authors present some properties and inequalities for the $p,q$-generalized psi-function. Also they obtain double inequalities bounding ratios of $q,k$ and $p,q$-generalized Gamma functions.

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1. Introduction

The classical Euler’s Gamma function is one of the most important special functions with applications in many fields such as analysis, mathematical physics, statistics and probability theory. In [4], Diaz and Truel introduced the $q,k$-generalized Gamma function, and also in [10], Krasniqi and Merovci defined the $p,q$-generalized Gamma function. This work is devoted to establish some properties and also inequalities concerning ratios of these generalized functions.

The paper is organized as follows: In next Section 2, we present some notations and preliminaries that will be helpful in the sequel. In Section 3 we give some properties and inequalities for the functions $\Gamma_{p,q}(x)$ and $\psi_{p,q}(x)$ for $x > 0$. Also, we present double inequalities involving a ratio of the functions $\Gamma_{q,k}(x)$ and $\Gamma_{p,q}(x)$.
2. Notations and Preliminaries

In this section, we present some definitions to make this paper self-containing. The reader can find details, e.g. in [3, 4, 5, 8, 11].

The well-known Euler’s Gamma function is defined by the following integral for \( x > 0 \),

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt,
\]

and it has also an equivalent limit expression as

\[
\lim_{n \to \infty} \frac{n! \, n^x}{x(x+1)(x+2) \ldots (x+n)},
\]

see [1, 2, 12]. The psi- or digamma-function, \( \psi(x) \), is defined as the logarithmic derivative of the Gamma function. That is,

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}
\]

for \( x > 0 \). The series representation is

\[
\psi(x) = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n + x)},
\]

where \( \gamma \) denotes Euler’s constant.

Diaz and Teruel [4] defined the \( q,k \)-generalized Gamma function \( \Gamma_{q,k}(x) \) for \( k > 0, q \in (0,1) \) and \( x > 0 \) by the formula

\[
\Gamma_{q,k}(x) = \frac{(1-q^k)^{\frac{x}{q,k}-1}}{(1-q)^{\frac{x}{q,k}-1}} = \frac{(1-q^x)^{\infty}_{q,k}}{(1-q)^{\infty}_{q,k} (1-q)^{\frac{x}{q,k}-1}},
\]

where

\[
(x+y)^n_{q,k} = \prod_{i=0}^{n-1} (x + q^k y), \quad (1+x)^\infty_{q,k} = \prod_{i=0}^{\infty} (1 + q^k x),
\]

\[
(1+x)^t_{q,k} = \frac{(1+xq^k)^t_{q,k}}{(1+q^k t x)_{q,k}^{\infty}}
\]

for \( x, y, t \in \mathbb{R} \) and \( n \in \mathbb{N} \) and \( \Gamma_{q,k}(x) \to \Gamma(x) \) as \( q \to 1 \) and \( k \to 1 \).

Also, Krasniqi and Merovci [10] defined the \( p,q \) extension of the Gamma function for \( p \in \mathbb{N}, q \in (0,1) \) and \( x > 0 \) as

\[
\Gamma_{p,q}(x) = \frac{[p]_q! [p]^x_q}{[x]_q [x+1]_q [x+2]_q \ldots [x+p]_q},
\]
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where $[p]_q = \frac{1-q^p}{1-q}$ and $\Gamma_{p,q}(x) \to \Gamma(x)$ as $p \to \infty$ and $q \to 1$.

The functions $\Gamma_{q,k}(x)$ and $\Gamma_{p,q}(x)$ satisfy the following identities:

$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x)$, $\Gamma_{q,k}(k) = 1$ and $\Gamma_{p,q}(x+1) = [x]_q \Gamma_{p,q}(x)$, $\Gamma_{p,q}(1) = 1$.

Similarly to the definition of $\psi(x)$, the $q,k$ and $p,q$-generalized of psi- (or digamma-) functions are defined respectively as:

$$\psi_{q,k}(x) = \frac{d}{dx} \ln \Gamma_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)},$$

$$\psi_{p,q}(x) = \frac{d}{dx} \ln \Gamma_{p,q}(x) = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}$$

for $x > 0$, and they satisfy the series representations

$$\psi_{q,k}(x) = -\frac{1}{k} \ln(1-q) + (\ln q) \sum_{n=0}^{\infty} \frac{q^{nk+x}}{1-q^{nk+x}},$$

$$\psi_{p,q}(x) = \ln[p]_q + (\ln q) \sum_{n=0}^{p} \frac{q^{n+x}}{1-q^{n+x}},$$

where $\psi_{q,k}(x) \to \psi(x)$ as $q \to 1$ and $k \to 1$, $\psi_{p,q}(x) \to \psi(x)$ as $p \to \infty$ and $q \to 1$, [7, 10].

The function $f$ is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds:

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y).$$

Note that the functions $\Gamma_{q,k}$ and $\Gamma_{p,q}$ are log-convex, [9, 10].

In the paper [6], the authors proved the inequality

$$\prod_{i=1}^{n} \frac{\Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)^{\lambda}} \leq \prod_{i=1}^{n} \frac{\Gamma_{q,k}(b_i)^{\mu_i}}{\Gamma_{q,k}(\beta)^{\lambda}}$$

by using the method based on some monotonicity properties of $q,k$-extension of the Gamma function.

In this paper, one of our aim is to establish a generalization of equation (3) by using techniques similar to those of [6].
3. Main Results

We now present the results of this paper. Let us begin with the following theorem.

**Theorem 1.** For $x > 0$, $p, n \in \mathbb{N}$ and $0 < q < 1$, the following inequality is valid:

$$\frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(x)} < [p]_q^{nx-1}.$$  \hspace{1cm} (4)

**Proof.** Using the definition of $\Gamma_{p,q}$ for $x$ and $nx$, we get

$$\frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(x)} = \frac{[p]_q^{nx}}{[p]_q^x} \cdot \frac{[nx]_q[x + 1]_q \ldots [x + p]_q}{[nx]_q[nx + 1]_q \ldots [nx + p]_q} < [p]_q^{nx-1},$$

and thus the result follows. \hfill \square

**Corollary 2.** The inequality

$$\Gamma_{p,q}(x + y) \leq [p]_q^{x+y-1} \sqrt{\Gamma_{p,q}(x) \Gamma_{p,q}(y)}$$

holds for $x, y > 0$, $p, n \in \mathbb{N}$ and $0 < q < 1$.

**Proof.** Since $\Gamma_{p,q}$ is log-convex, we can write

$$\Gamma_{p,q}(\frac{x + y}{2}) \leq \sqrt{\Gamma_{p,q}(x) \Gamma_{p,q}(y)}.$$  \hspace{1cm} (5)

Then

$$\Gamma_{p,q}(x + y) \leq \sqrt{\Gamma_{p,q}(2x) \Gamma_{p,q}(2y)}.$$

From equation (4) in the last theorem we get for $n = 2$ that

$$\Gamma_{p,q}(2x) \leq \Gamma_{p,q}(x)[p]_q^{2x-1}, \quad \Gamma_{p,q}(2y) \leq \Gamma_{p,q}(y)[p]_q^{2y-1}.$$

Hence we get the result. \hfill \square

The $p, q$-extension of the psi-function is similarly defined as

$$\psi_{p,q}(x) = \frac{d}{dx} \ln \Gamma_{p,q}(x) = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}.$$
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It satisfies the series representation:

$$\psi_{p,q}(x) = \ln[p]_q + (\ln q) \sum_{n=0}^{p} \frac{q^{n+x}}{1 - q^{n+x}},$$

(6)

where $\psi_{p,q}(x) \to \psi(x)$ as $p \to \infty$ and $q \to 1$, [6].

**Lemma 3.** For $x > 0$, $p \in \mathbb{N}$ and $0 < q < 1$, the function $\psi_{p,q}(x)$ satisfies the equation:

$$\psi_{p,q}(x + 1) = -\ln q \frac{q^x}{1 - q^x} + \psi_{p,q}(x).$$

(7)

**Proof.** Since

$$\Gamma_{p,q}(x + 1) = [x]_q \Gamma_{p,q}(x),$$

(8)

by differentiating with respect to $x$ both parts of equation (8), it follows:

$$\frac{d}{dx} \Gamma_{p,q}(x + 1) = -\ln q \frac{q^x}{1 - q^x} \Gamma_{p,q}(x) + [x]_q \frac{d}{dx} \Gamma_{p,q}(x).$$

(9)

By dividing both parts of (9) by $\Gamma_{p,q}(x)$, taking in mind the definition of $\psi_{p,q}(x)$ and equation (8), we obtain the desired equation. \hfill \square

**Remark 4.** By induction and using

$$\Gamma_{p,q}(x + 1) = [x]_q \Gamma_{p,q}(x),$$

we get

$$\Gamma_{p,q}(x + n) = [x]_{n,q} \Gamma_{p,q}(x)$$

for $x > 0$, $p \in \mathbb{N}$, $0 < q < 1$ and $n \in \mathbb{N}$ where

$$[x]_{n,q} = [x]_q [x + 1]_q [x + 2]_q \cdots [x + (n - 1)]_q.$$

**Theorem 5.** The function $\psi_{p,q}(x)$ satisfies the recurrence formula

$$\psi_{p,q}(x + n) = \psi_{p,q}(x) - \ln q \sum_{j=0}^{n-1} \frac{q^{x+j}}{1 - q^{x+j}}$$

for $x > 0$, $p \in \mathbb{N}$ and $0 < q < 1$. 

Proof. The equality will be proved by induction. For $n = 1$ it holds, because of equation (7). We suppose that our assumption holds for $n$ and we will prove that it holds also for $n + 1$.

Since we have

$$
\psi_{p,q}(x + (n + 1)) = \psi_{p,q}(x + n) + 1
$$

$$
= \psi_{p,q}(x + n) - \ln q \frac{q^{x+n}}{1 - q^{x+n}}
$$

$$
= \psi_{p,q}(x) - \ln q \sum_{j=0}^{n-1} \frac{q^{x+j}}{1 - q^{x+j}} - \ln q \frac{q^{x+n}}{1 - q^{x+n}}
$$

$$
= \psi_{p,q}(x) - \ln q \sum_{j=0}^{n} \frac{q^{x+j}}{1 - q^{x+j}},
$$

then our assumption is true for every $n \in \mathbb{N}$. Hence the result follows.

Theorem 6. The following inequalities are valid for $x > 0$, $p \in \mathbb{N}$ and $0 < q < 1$:

$$
\frac{q^x}{1 - q^x} \ln q + \ln[x]_q < \psi_{p,q}(x) < \ln[x]_q.
$$

(10)

Proof. Let $f(x) = \ln \Gamma_{p,q}(x)$. We apply the mean value theorem to this function in the interval $(x, x + 1)$.

Then, there is $x_0 \in (x, x + 1)$ such that the equality

$$
\ln \Gamma_{p,q}(x + 1) - \ln \Gamma_{p,q}(x) = \psi_{p,q}(x_0)
$$

holds, and using

$$
\Gamma_{p,q}(x + 1) = [x]_q \Gamma_{p,q}(x)
$$

we get

$$
\psi_{p,q}(x_0) = \ln[x]_q.
$$

Since

$$
\psi'_{p,q}(x) = \ln^2 q \sum_{k=0}^{p} \frac{q^{x+k}}{(1 - q^{x+k})^2} > 0,
$$

we have $\psi_{p,q}(x)$ is increasing on $(0, \infty)$. Then we obtain

$$
\psi_{p,q}(x) < \psi_{p,q}(x_0) < \psi_{p,q}(x + 1).
$$
Since we got
\[ \psi_{p,q}(x + 1) = -\ln q \frac{q^x}{1 - q^x} + \psi_{p,q}(x), \]
we have
\[ \psi_{p,q}(x) < \ln[x]_q < -\ln q \frac{q^x}{1 - q^x} + \psi_{p,q}(x), \]
and the result follows.

**Corollary 7.** For \( p \in \mathbb{N} \) and \( 0 < q < 1 \) we have
\[ \frac{q}{1 - q} \ln q < \psi_{p,q}(1) < 0 \]
and for \( x \in (0, 1] \) we have
\[ \psi_{p,q}(x) < 0. \]

**Lemma 8.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing function on any open interval and \( \alpha, \beta, \gamma_i, \mu, \lambda, b \) be real numbers such that
\[ b + \alpha x \leq \beta + \sum_{i=1}^{n} \gamma_i x, \ \gamma_i \lambda \geq \alpha \mu > 0. \]
If
\[ f(b + \alpha x) > 0 \text{ or } f(\beta + \sum_{i=1}^{n} \gamma_i x) > 0, \]
then
\[ \alpha \mu f(b + \alpha x) - \lambda \gamma_i f(\beta + \sum_{i=1}^{n} \gamma_i x) \leq 0 \]
(11)
is valid.

**Proof.** Let \( f(b + \alpha x) > 0. \) Since \( f \) is increasing, \( f(b + \alpha x) \leq f(\beta + \sum_{i=1}^{n} \gamma_i x). \)
Then \( f(\beta + \sum_{i=1}^{n} \gamma_i x) > 0. \)
Writing
\[ \alpha \mu f(b + \alpha x) \leq \alpha \mu f(\beta + \sum_{i=1}^{n} \gamma_i x) \leq \lambda \gamma_i f(\beta + \sum_{i=1}^{n} \gamma_i x), \]
leads us to equation (11). This time, let $f(\beta + \sum_{i=1}^{n} \gamma_i x) > 0$. Then $f(b + \alpha x) > 0$ or $f(b + \alpha x) \leq 0$.

If $f(b + \alpha x) > 0$, then the proof is completed. And if $f(b + \alpha x) \leq 0$; since $\gamma_i \lambda \geq \alpha \mu > 0$ we have

$$\gamma_i \lambda f(b + \alpha x) \leq \alpha \mu f(\beta + \sum_{i=1}^{n} \gamma_i x) \leq \gamma_i \lambda f(\beta + \sum_{i=1}^{n} \gamma_i x).$$

Hence equation (11) holds. \qed

One can prove the following lemma immediately:

**Lemma 9.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function on any open interval and $\alpha, \beta, \gamma_i, \mu, \lambda, b$ be real numbers such that

$$b + \alpha x \leq \beta + \sum_{i=1}^{n} \gamma_i x, \ \alpha \mu \geq \gamma_i \lambda > 0.$$ 

Then if

$$f(b + \alpha x) < 0 \text{ or } f(\beta + \sum_{i=1}^{n} \gamma_i x) < 0,$$

the inequality (11) still holds.

**Preparation for Applications:**

Since $\psi_{q,k}(x)$ and $\psi_{p,q}(x)$ are increasing functions on the open interval $(0, \infty)$, we can write $\psi_{q,k}(x)$ or $\psi_{p,q}(x)$ in equation (11) instead of $f$.

**Applications to the $q,k$ Generalized Gamma Function:**

We apply Lemmas 8 and 9 to the function $\Gamma_{q,k}$. Note that one can get similar results for the generalized $p,q$-Gamma function $\Gamma_{p,q}$.

**Theorem 10.** Let $\alpha_i, \beta, \gamma_i, \mu_i, \lambda, b_i$ be positive real numbers such that

$$b_i + \alpha_i x \leq \beta + \sum_{i=1}^{n} \gamma_i x, \ \gamma_i \lambda \geq \alpha_i \mu_i > 0.$$ 

If

$$\psi_{q,k}(b_i + \alpha_i x) > 0 \text{ or } \psi_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x) > 0,$$
then
\[ g(x) = \frac{n \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x)^{\lambda}} \]
is decreasing function for \( x \geq 0 \).

**Proof.** Let \( H(x) = \ln g(x) \). Then,
\[
H(x) = \ln \frac{n \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x)^{\lambda}} = \mu_i \ln \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x) - \lambda \ln \prod_{i=1}^{n} \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x).
\]
We have
\[
H'(x) = \sum_{i=1}^{n} \mu_i \alpha_i \psi_{q,k}(b_i + \alpha_i x) \frac{\Gamma_{q,k}(b_i + \alpha_i x)}{\Gamma_{q,k}'(b_i + \alpha_i x)} - \lambda \sum_{i=1}^{n} \frac{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x)}{\Gamma_{q,k}'(\beta + \sum_{i=1}^{n} \gamma_i x)}
\]
\[
= \sum_{i=1}^{n} \left[ \mu_i \alpha_i \psi_{q,k}(b_i + \alpha_i x) - \lambda \gamma_i \psi_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x) \right] \leq 0.
\]
This implies that \( H \) is decreasing on \( x \in [0, \infty) \). As a result, \( g \) is decreasing on \( x \in [0, \infty) \). □

**Corollary 11.** Let \( \alpha_i, \beta, \gamma_i, \mu_i, \lambda, b_i \) be positive real numbers such that
\[ b_i + \alpha_i x \leq \beta + \sum_{i=1}^{n} \gamma_i x, \ \gamma_i \lambda \geq \alpha_i \mu_i > 0 \]
and let
\[ \psi_{q,k}(b_i + \alpha_i x) > 0 \ or \ \psi_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x) > 0. \]
Then for \( x \in [0, 1] \) we have
\[
\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i} \leq \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x) \leq \prod_{i=1}^{n} \Gamma_{q,k}(b_i)^{\mu_i},
\]
and for \( x \in [1, \infty) \) we have
\[
\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i} \leq \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i)^{\mu_i} \leq \prod_{i=1}^{n} \Gamma_{q,k}(b_i)^{\mu_i}.
\]

Proof. Since \( g(x) = \prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i} \) is decreasing function, for \( x \in [0, 1] \) we have
\[
g(1) \leq g(x) \leq g(0),
\]
and for \( x \in [1, \infty) \)
\[
g(x) \leq g(1);
\]
yielding the results.

Remark 12. Let \( \alpha, \beta, \gamma_i, \mu, \lambda, b \) be real numbers such that
\[b + \alpha x \leq \beta + \sum_{i=1}^{n} \gamma_i x, \quad \alpha \mu \leq \gamma_i \lambda > 0.\]
Then if
\[\psi_{q,k}(b + \alpha x) < 0 \quad \text{or} \quad \psi_{q,k}(\beta + \sum_{i=1}^{n} \gamma_i x) < 0,\]
inequalities (12) and (13) are hold.

Remark 13. If we set \( \gamma_i = \alpha_i \) in Theorem 10 and Corollary 11, we obtain inequalities (3.3) and (3.4) from [6].
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References


