IDEALS IN SEMIRING WITH INVOLUTION

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Abstract: In this paper, we study the notion of $*$-prime ideal in semiring with involution and shown that if $M$ is a non-void $*$-$m$-system in a semiring with involution and if $I$ is a $*$-ideal of $R$ with $I \cap M = \phi$, then there exists a $*$-prime ideal $P$ of $R$ such that $I \subseteq P$ and $P \cap M = \phi$. We also introduce the notion of $*$-$k$-prime ideal and we have shown that if $P$ is a $*$-$k$-ideal of a semiring $R$ with involution, then $P$ is semiprime if and only if $P$ is $*$-$k$-prime.

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1. Introduction

The concept of semirings was introduced by H.S. Vandiver in 1935, and it has been studied by several authors. Throughout this paper $R$ denotes a semiring. A semiring $R$ is a non-empty set $R$ together with two binary operation $+$ and $\cdot$ such that:

i) $< R, + >$ is a commutative monoid with identity denoted by $0_R$ or simply $0$,
ii) \( < R, \cdot, > \) is a semigroup,

iii) For every \( r, s, t \in R \), \( r(s + t) = rs + rt \) and \( (s + t)r = sr + tr \),

iv) For every \( r \in R \), \( r0 = 0r = 0 \).

Recall from [3] that a semiring with involution is an algebra \( R = < R, +, \cdot, * > \) such that \( < R, +, \cdot, > \) is a semiring, and the following identities are satisfied:

\[
(a + b)^* = a^* + b^*; (ab)^* = b^*a^*; (a^*)^* = a.
\]

For any nonempty set \( S \), we define \( S^* = \{ s^* : s \in S \} \). Observe that involution of every non-zero element is non-zero. A non-empty subset \( I \) of a semiring \( R \) is called a left (resp. right) ideal of \( R \) if \( a + b \in I \), \( ra \in I \) (resp. \( ar \in I \)) for all \( a, b \in I \) and for all \( r \in R \). If \( I \) is both left and right ideal of \( R \), then \( I \) is called an ideal of \( R \). Following [2], we say that an ideal \( I \) of \( R \) is said to be \(*\)-ideal if \( I^* \subseteq I \). Clearly if \( I \) is \(*\)-one sided ideal of \( R \), then \( I \) is a \(*\)-ideal of \( R \). Observe that if \( K \) is an ideal of \( R \), then \( K^*K \), \( KK^* \), \( K \cap K^* \) and \( K + K^* \) are \(*\)-ideals of \( R \) and \( K^* \) is also an ideal of \( R \). An ideal \( P \) is said to be prime if whenever \( A, B \) are ideals of \( R \) such that \( AB \subseteq P \), then \( A \subseteq P \) or \( B \subseteq P \). Following [2], we say that a \(*\)-ideal \( P \) of \( R \) is said to be \(*\)-prime if whenever \( A, B \) are \(*\)-ideals of \( R \) such that \( AB \subseteq P \), then \( A \subseteq P \) or \( B \subseteq P \). Observe that if \( P \) is a prime and \(*\)-ideal of \( R \), then \( P \) is a \(*\)-prime ideal of \( R \). The following example shows that there exists a \(*\)-prime ideal of \( R \) which is not prime.

**Example 1.1.** Consider the ring \( Z_6 \) and commutative semiring \( B = B(3, 2) \) (F.E. Alarcon and D. Polkoska [1]).

Let \( R = Z_6 \oplus Z_6 \oplus B \) be a semiring. Define \(*\)-on \( R \) via \((a_1,a_2,b_1)^* = (a_2,a_1,b_1)\). Let \( A = \{0, 3\} \). Then \( P = (A, A, B) \) is a \(*\)-prime ideal but not a prime ideal, since if \( I = \{0, 2, 4\} \) and if \( B = (I, 0, 0) \) and \( C = (0, I, 0) \), then \( BC \subseteq P \) but neither \( B \) nor \( C \) is included in \( P \) and hence \( P \) is not prime.

But the notion of \(*\)-semiprime ideal and semiprime ideal are coincide. Indeed, if \( I \) is a \(*\)-semiprime ideal of \( R \) and \( J \) is an ideal of \( R \) with \( J^2 \subseteq I \). Then \((J^*)^2 \subseteq I \) and \((J + J^*)^2 = J^2 + JJ^* + J^*J + (J^*)^2 \subseteq J + I \) which imply \((J + J^*)^4 \subseteq I \), so \( J \subseteq J + J^* \subseteq I \).

In 1956, M. Henriksen [4] defined a more restricted class of ideals in semirings, which he called \( k \)-ideal. A left (resp. right) \( k \)-ideal \( I \) of \( R \) is called left (resp. right) \( k \)-ideal if \( a \in I \) and \( x \in R \) and if \( a + x \in I \), then \( x \in I \). If \( I \) is both left and right \( k \)-ideal of \( R \), then \( I \) is \( k \)-ideal of \( R \). Clearly intersection of \( k \)-ideals of \( R \) is again \( k \)-ideal of \( R \) and \( I \) is a \( k \)-ideal of \( R \) if and only if \( I^* \) is a \( k \)-ideal of \( R \). A \(*\)-\( k \)-ideal \( I \) is a \( k \)-ideal and \( I^* \subseteq I \). If \( I \) is a \( k \)-ideal of \( R \), then \( I \cap I^* \) is a \(*\)-\( k \)-ideal of \( R \). For subsets \( A, B \) of \( R \), we denote \((A : B)_l = \{ r \in R/rB \subseteq A \}\) and \((A : B)_r = \{ r \in R/Br \subseteq A \}\). For
any \( a \in R \), \(< a >\) the principle ideal of \( R \) generated by \( a \). One can easily prove that \(< a > = \{ na + sa + at + \sum_i s_i at_i / n \in N^+, s, t, s_i, t_i \in R \} \) and \(< a^* > = < a >^* \).

2. Main Results

**Lemma 2.1.** Let \( R \) be a semiring.

i) If \( A \) and \( B \) are left (resp. right) ideals of \( R \), then \((A : B)_l\) (resp. \((A : B)_r\)) is an ideal of \( R \).

ii) If \( A \) and \( B \) are left (resp. right)- \( k \)-ideal of \( R \), then \((A : B)_l\) (resp. \((A : B)_r\)) is a \( k \)-ideal of \( R \).

**Lemma 2.2.** Let \( R \) be a semiring with involution and let \( P \) be a \( * \)-ideal of \( R \). Then \( P \) is a \( * \)-prime ideal of \( R \) if and only if whenever \( AB \subseteq P \), we have \( A \subseteq P \) or \( B \subseteq P \) with either \( A \) or \( B \) is a \( * \)-ideal.

**Proof.** Let \( P \) be a \( * \)-prime ideal of \( R \). Without loss of generality, let us assume that \( A \) is an ideal of \( R \) and \( B \) is a \( * \)-ideal of \( R \) such that \( AB \subseteq P \). Then \( BA^* \subseteq P \) and \((A^*B)^2 = A^*BA^*B \subseteq P \) which imply \( AB \subseteq P \). Thus \((A + A^*)B \subseteq P \). By assumption, we have \((A + A^*) \subseteq P \) or \( B \subseteq P \). Hence \( A \subseteq P \) or \( B \subseteq P \). The converse is obvious.

**Theorem 2.3.** Let \( R \) be a semiring with involution and \( P \) be a \( * \)-ideal of \( R \). Then the following conditions are equivalent:

(i) \( P \) is a \( * \)-prime ideal.

(ii) If \( a, b \in R \) such that \( aRb \subseteq P \); \( a^*Rb \subseteq P \), then \( a \in P \) or \( b \in P \).

(iii) If \(< a > \) and \(< b > \) are principal ideals of \( R \) such that \(< a > < b > \subseteq P \); \(< a^* > < b > \subseteq P \), then \( a \in P \) or \( b \in P \).

(iv) If \( U \) and \( V \) are right ideals in \( R \) such that \( UV \subseteq P \); \( U^*V \subseteq P \), then \( U \subseteq P \) or \( V \subseteq P \).

(v) If \( U \) and \( V \) are left ideals in \( R \) such that \( UV \subseteq P \); \( U^*V \subseteq P \), then \( U \subseteq P \) or \( V \subseteq P \).

**Proof.** \((i) \Rightarrow (ii)\) Suppose \( aRb \subseteq P \) and \( a^*Rb \subseteq P \). By Lemma 2.1, we have \(< a > < b > \subseteq P \) and \(< a^* > < b > \subseteq P \). Then \( R(< a > + < a^* >)RR < b > R \subseteq P \). Then \( R(< a > + < a^* >)RR < b > R \subseteq P \). By Lemma 2.2, we have \( R(< a > + < a^* >) \subseteq P \) or \( R < b > R \subseteq P \).

If \( R(< a > + < a^* >)R \subseteq P \), then \(< a > + < a^* >)^3 \subseteq P \). Hence \( a \in P \).
Otherwise $R < b > R \subseteq P$. Then $< b >^3 \subseteq P$ implies $b \in < b > \subseteq P$.

(iii) $\Rightarrow$ (iii) It is obvious.

(v) Let $U$ and $V$ be left ideals of $R$ such that $UV \subseteq P$ and $U \ast V \subseteq P$. Suppose $U \not\subseteq P$. Then there exists $u \in U$ such that $u \notin P$. Let $v \in V$. Then $< u > < v > \subseteq UV + RUV \subseteq P$ and $< u > < v > \subseteq U \ast V + RU \ast V \subseteq P$. By assumption, we have $< u > \subseteq P$ or $< v > \subseteq P$, but $u \notin P$. Hence $V \subseteq P$.

\[\square\]

A non-empty set $M$ of elements of a semiring $R$ is said to be $m$-system if $a, b \in M$, there exists $x \in R$ such that $axb \in M$. A non-empty set $M$ of elements of a semiring $R$ is said to $\ast$-$m$-system if $a, b \in M$, there exists $x \in R$ such that $axb \in M$ or $a \ast xb \in M$. Obviously every $m$-system is a $\ast$-$m$-system. Also $P$ is a $\ast$-prime ideal if and only if its complement is a $\ast$-$m$-system.

The following example shows that there exists a $\ast$-$m$-system of $R$ that is not an $m$-system of $R$.

**Example 2.4.** Let $R$ be a semiring of non-negative integers where $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$. Let $Z_6$ be the ring of integer of modulo 6. Then $S = Z_6 \oplus Z_6 \oplus R \oplus R$ is a semiring. Define $\ast$-on $R$ via $(a_1, a_2, b_1, b_2)\ast = (a_2, a_1, b_2, b_1)$.

Let $M = \{(i, j, m, n) / i \neq j; i, j \neq 0; m, n < 3\}$. Clearly $M$ is a $\ast$-$m$-system but not a $m$-system because $(2, 3, 1, 2)x(3, 2, 1, 2) \notin M$ for all $x \in S$.

**Theorem 2.5.** Let $M$ be a non-void $\ast$-$m$-system in $R$ and $I$ be a $\ast$-ideal of $R$ with $I \cap M = \phi$. Then $I$ is contained in a $\ast$-prime ideal $P \neq R$ with $P \cap M = \phi$.

**Proof.** Let $A = \{J / J$ is a $\ast$-ideal of $R$ with $I \subseteq J$ and $J \cap M = \phi\}$. Clearly $A \neq \phi$. By Zorn’s lemma, $A$ contains a maximal element (say) $P$ with $P \subseteq I$ and $P \cap M = \phi$. Let $A, B$ be $\ast$-ideals of $R$ such that $AB \subseteq P$. Suppose $A \not\subseteq P$ and $B \not\subseteq P$. Then there exists $a \in A$ and $b \in B$ such that $a, b \notin P$. Now $P \subset P + (a > + < a >^\ast)$ and $P \subset P + (b > + < b >^\ast)$ which gives $(P + (a > + < a >^\ast)) \cap M \neq \phi$ and $(P + (b > + < b >^\ast)) \cap M \neq \phi$. Then there exists $x \in (P + (a > + < a >^\ast)) \cap M$ and $y \in (P + (b > + < b >^\ast)) \cap M$ such that $xty \in M$ or $x^t y \in M$ for some $t \in R$. Clearly $xty \in (P + (a > + < a >^\ast))(P + (b > + < b >^\ast))$ and $x^t y \in (P + (a > + < a >^\ast))(P + (b > + < b >^\ast))$. Now $(P + (a > + < a >^\ast))(P + (b > + < b >^\ast)) \subseteq P + (a > + < a >^\ast < b > + < a >^\ast < b >^\ast) \subseteq P + AB \subseteq P$. Then
$P \cap M \neq \phi$, a contradiction. Hence $P$ is a $*$-prime ideal of $R$ contains $I$. \hfill \Box

3. $*$-$k$-Prime Ideal

In this section, we continue our investigation of interrelations between various types of ideals in semiring with involution. Also, we introduce the notions of $*$-$k$-prime and $*$-$m_k$-system.

From [7], if $I$ is any additive subsemigroup of $R$, then $\overline{I} = \{ a \in R \mid a + x \in I \text{ for some } x \in I \}$ is called $k$-closure of $I$. Observe that $I \subseteq \overline{I}, \overline{I} = \overline{I}$ and $\overline{I} = \overline{I}$. It is easy to verify that if $I$ is an ideal of $R$, then $I$ is $k$-ideal if and only if $I = \overline{I}$. If $I$ is an ideal of $R$, then $\overline{I}$ is an ideal of $R$. Observe that $\langle a \rangle$ is a principal $k$-ideal generated by $a$. Following [5], an ideal $P$ is said to be $*$-$k$-prime if whenever $A, B$ are $k$-ideals of $R$ such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. A $*$-ideal $P$ of $R$ is said to be $*$-$k$-prime if whenever $A, B$ are $*$-ideals of $R$ such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. From [5], a non-empty set $M$ of elements of a semiring $R$ is said to be $m_k$-system if $a, b \in M$, there exists $x \in \langle a \rangle$ and $y \in \langle b \rangle$ such that $xy \in M$. A non-empty set $M$ of elements of a semiring $R$ is said to be $*-m_k$-system if $a, b \in M$, there exists $x \in \langle a \rangle + \langle a^* \rangle$ and $y \in \langle b \rangle + \langle b^* \rangle$ such that $xy \in M$ or $x^*y \in M$. Observe that every $m_k$-system is a $*-m_k$-system. In Example 2.4 $M$ is a $*-m_k$-system not an $m_k$-system. It is easy to see that if $P$ is a $*$-ideal in $R$, then $P$ is $*$-$k$-prime if and only if $R/P$ is $*$-$m_k$-system. Also if $P$ is an ideal of $R$, then $P$ is $k$-prime if and only if $R/P$ is an $m_k$-system. Let $I$ be a additive subsemigroup of $R$ and let $L(I) = \{ x \in I \mid Rx \subseteq I \}$ and $H(I) = \{ y \in L(I) \mid yR \subseteq L(I) \}$. Clearly $L(I)$ is a left ideal of $R$.

Lemma 3.1. Let $R$ be a semiring. If $I$ is any additive subsemigroup of $R$, then $H(I)$ is the (unique) largest ideal of $R$ contained in $I$.

Proof. Clearly $H(I) \subseteq I$. From [10, Proposition 4], we have $H(I)$ is the largest ideal of $R$ contained in $I$. \hfill \Box

It is well-known [7] that if $I$ is an ideal of $R$, then $\overline{I}$ is the smallest $k$-ideal containing $I$.

Lemma 3.2. Let $R$ be a semiring. If $I$ is a additive subsemigroup of $R$ with $I = \overline{I}$, then $I$ is an $k$-ideal of $R$ or $H(I)$ is a $k$-ideal of $R$ and it is the largest $k$-ideal contained in $I$. 


Proof. By Lemma 3.1, we have $H(I)$ is the largest ideal of $R$ contained in $I$. Clearly $H(I) \subseteq H(I)$ and $H(I)$ is an ideal of $R$. Let $x \in H(I)$. Then $x + h \in H(I)$ for some $h \in H(I)$. Since $H(I) \subseteq I$, we have $x + h \in I$ for some $h \in I$. Since $I = \overline{I}$, we have $x \in I$. Thus $H(I) \subseteq I$. By Lemma 3.1, we have $H(I) = I$ or $H(I) = \overline{H(I)}$. Hence $I$ is a $k$-ideal of $R$ or $H(I)$ is a $k$-ideal of $R$.

**Theorem 3.3.** Let $R$ be a semiring and let $P$ be a $k$-ideal of $R$. Then $P$ is a prime ideal if and only if $P$ is a $k$-prime ideal.

Proof. If $P$ is a prime then $P$ is $k$-prime. Let $A$ and $B$ be ideals of $R$ such that $AB \subseteq P$. From Lemma 2.1, we have $\overline{A} \overline{B} \subseteq P$. Then by assumption, we have $A \subseteq P$ or $B \subseteq P$. Hence $P$ is a prime ideal. □

**Theorem 3.4.** Let $R$ be a semiring with involution and $P$ be a $*-k$-ideal of $R$. Then $P$ is $*-prime$ if and only if $P$ is $*-k$-prime.

**Theorem 3.5.** Let $R$ be a semiring with involution and let $P$ be a $*-k$-ideal of $R$. Then $P$ is semiprime if and only if $P$ is $*-k$-semiprime.

Proof. If $P$ is a semiprime ideal, then clearly $P$ is $*-k$-semiprime.

Conversely, let $P$ be a $*-k$-semiprime ideal, and let $J$ be any ideal of $R$ with $J^2 \subseteq P$. Also $(J^*)^2 \subseteq P$. Then $(J + J^*)^4 \subseteq P$. Since $(P : (J + J^*)^2)_l$ and $(P : (J + J^*)^2)_r$ are $k$-ideals of $R$, we have $(J + J^*)^2 \subseteq P$. By assumption, we have $(J + J^*)^2 \subseteq P$. Then $(J + J^*)^2 \subseteq P$. Again by using $(P : (J + J^*))_l$ and $(P : (J + J^*))_r$, we have $(J + J^*)^2 \subseteq P$. Then $J + J^* \subseteq P$. Thus $J \subseteq P$. Hence $P$ is a semiprime ideal. □

**Lemma 3.6.** Let $R$ be a semiring with involution. If $P$ is a $k$-prime and $*-ideal$ of $R$, then $P$ is $*-k$-prime.

The converse of Lemma 3.6 is not true, in general as the following example shows.

**Example 3.7.** Consider the ring $A = Z_4$ of modulo 4 and semiring $B = B(4, 2)$ (F.E. Alarcon and D. Polkoska [5]).
Here \( R = A \oplus A \oplus B \oplus B \) is a semiring. Define \(*\)-on \( R \) via \((a_1, a_2, b_1, b_2)^* = (a_2, a_1, b_2, b_1)\). Let \( A_1 = \{0, 2\} \), \( P = (A, A, A_1) \), \( I = (A, A, B, 0) \) and \( J = (A, A, 0, B) \). Then \( P \) is a \(*\)-\( k \)-prime ideal of \( R \) but not a \( k \)-prime ideal because of \( IJ \subseteq P \) but neither \( I \subseteq P \) nor \( J \subseteq P \).

**Theorem 3.8.** Let \( R \) be a semiring with involution and let \( P \) be a \(*\)-\( k \)-ideal of \( R \). Then \( P \) is a \(*\)-\( k \)-prime ideal if and only if whenever \( AB \subseteq P \), we have \( A \subseteq P \) or \( B \subseteq P \) with either \( A \) or \( B \) is a \(*\)-\( k \)-ideal of \( R \).

**Proof.** Let \( P \) be a \(*\)-\( k \)-prime ideal of \( R \). Without loss of generality, let us assume that \( A \) is a \(*\)-\( k \)-ideal of \( R \) and \( B \) is an ideal of \( R \) and \( AB \subseteq P \). Then \( B^*A \subseteq P \). Thus \((AB^*)^2 \subseteq P \). By Theorem 3.5, we have \( AB^* \subseteq P \). Then \( A(B + B^*) \subseteq P \). Since \((P : A)_r \) is a \( k \)-ideal of \( R \), we have \( A(B + B^*) \subseteq P \). By assumption, we have \( A \subseteq P \) or \((B + B^*) \subseteq P \). Hence \( A \subseteq P \) or \( B \subseteq P \). Converse is clear. \( \square \)

**Theorem 3.9.** Let \( Q \) be a \(*\)-ideal of a semiring \( R \) with involution and let \( M \) be a \(*\)-\( m_k \)-system of \( R \) such that \( Q \cap M = \phi \). Then there exists a \(*\)-prime ideal \( P \neq R \) such that \( Q \subseteq P \) with \( P \cap M = \phi \).

**Proof.** Let \( A = \{ J / J \text{ is } *-\text{ideal of } R \text{ such that } Q \subseteq J \text{ and } J \cap M = \phi \} \). Clearly \( A \neq \phi \). By Zorn’s Lemma, \( A \) contains a maximal element (say) \( P \) with \( Q \subseteq P \) and \( P \cap M = \phi \). Let \( A \) and \( B \) be \(*\)-ideals of \( R \) such that \( AB \subseteq P \). Suppose \( A \not\subseteq P \) and \( B \not\subseteq P \). Then there exists \( a \in A \) and \( b \in B \) with \( a, b \not\in P \). Thus \( P \subseteq P+ \langle a \rangle \cup \langle a^* \rangle \) and \( P \subseteq P+ \langle b \rangle \cup \langle b^* \rangle \). By maximality of \( P \), we have \( P+ \langle a \rangle \cup \langle a^* \rangle \cap M \neq \phi \) and \( P+ \langle b \rangle \cup \langle b^* \rangle \cap M \neq \phi \). Then there exists \( x \in \overline{P+ \langle a \rangle} \cup \langle a^* \rangle \) and \( y \in \overline{P+ \langle b \rangle} \cup \langle b^* \rangle \) such that \( x_1y_1 \in M \) or \( x_1y_1 \in M \) for some \( x_1 \in \langle x \rangle \cup \langle x^* \rangle \) and \( y_1 \in \langle y \rangle \cup \langle y^* \rangle \). Since \( x \in \overline{P+ \langle a \rangle} \cup \langle a^* \rangle \) and \( y \in \overline{P+ \langle b \rangle} \cup \langle b^* \rangle \), we have

\[
x_1y_1 \in (\overline{P+ \langle a \rangle} \cup \langle a^* \rangle)(\overline{P+ \langle b \rangle} \cup \langle b^* \rangle)
\]

and

\[
x_1^*y_1 \in (\overline{P+ \langle a \rangle} \cup \langle a^* \rangle)(\overline{P+ \langle b \rangle} \cup \langle b^* \rangle).
\]
Let \( s \in (P+<a>+<a^*>)(P+<b>+<b^*>). \) Then \( s = \sum_{i=1}^{n} t_i t'_i \) for some \( t_i \in P+<a>+<a^*> \) and \( t'_i \in P+<b>+<b^*> \). Thus \( t_i + x_i \in (P+<a>+<a^*>) \) and \( t'_i + x'_i \in (P+<b>+<b^*>) \) for \( x_i \in (P+<a>+<a^*>) \) and \( x_i \in (P+<b>+<b^*>) \) for each \( i \). Clearly \( (P+<a>+<a^*>)(P+<b>+<b^*>) \subseteq P \) and \( x_i x'_i \in P \subseteq \overline{P} \). Now Consider \( x_i t_i + x_i x'_i = x_i (t'_i + x'_i) \in (P+<a>+<a^*>)(P+<b>+<b^*>) \subseteq P \). Then \( x_i t_i \in \overline{P} \) since \( x_i x'_i \in P \). Similarly, we can get \( t_i t'_i \in \overline{P} \).

Since \( \overline{P} \) is an ideal of \( R \), we have \( t_i x_i + x_i t'_i + x_i x'_i \in \overline{P} \). Now \( t_i t'_i + x_i t'_i + t_i x'_i + x_i x'_i = (t_i + x_i)(t'_i + x'_i) \in (P+<a>+<a^*>)(P+<b>+<b^*>) \subseteq P \). Thus \( s \in \overline{P} \). Hence \( (P+<a>+<a^*>)(P+<b>+<b^*>) \subseteq \overline{P} \). So \( x_1 y_1 \) and \( x'_1 y'_1 \in \overline{P} \), a contradicts to \( \overline{P} \cap M = \phi \). Hence \( P \) is a \( \ast \)-prime ideal of \( R \) contains \( Q \).

\[ \square \]

**Theorem 3.10.** Let \( Q \) be a \( \ast \)-ideal of semiring with involution of \( R \), and let \( M \) be a \( \ast \)-\( m_k \)-system of \( R \) such that \( \overline{Q} \cap M = \phi \). Then there exists a \( \ast \)-\( k \)-prime ideal \( P \neq R \) such that \( Q \subseteq P \) with \( \overline{P} \cap M = \phi \).

**References**


