SOME GRAPHS WITH SUPER VERTEX SUM NUMBER 2

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Abstract: In this paper we give optimal super vertex sum labeling scheme for super subdivision of bi-star, path union of spider and algorithm to construct super vertex sum labeling of super subdivision of Caterpillar.

AMS Subject Classification: 05C78, 05C99, 11Z05
Key Words: sum labeling, sum number, sum graph, isolates

1. Introduction

We considered only simple, finite and undirected graphs for the study. For all terminologies and notations which are not elucidated in detail, we refer [3] and [2]. The sum labeling of graphs was introduced by Harary [4]. Let $G(p,q)$ be a graph and $f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\}$ be a bijective mapping. For every pair of adjacent vertices $u,v \in V(G)$ let $f(u) + f(v) = f(w)$ for some vertex $w \in V(G)$. Let $\mu_f(G)$ be the maximum of $f(u)$ for vertex $\forall u \in V(G)$. If $\mu_f(G) = |V(G)|$, then $f$ is called super vertex sum labeling, [5].

A graph that admits super vertex sum labeling will be always disconnected. This is because we need isolated vertices to super vertex sum label a graph. The least number of isolates needed to super vertex sum label the graph $G$ is called...
super vertex sum number of the graph, denoted by $\sigma_{sv}(G)$. It is proved that the lower bound of $\sigma_{sv}(G)$ is $\delta(G)$, [5]. It is also shown that super subdivision of path, cycles, stars and spiders are super vertex sum graph with $\sigma_{sv}(G) = 2$ [5].

The labeling with minimum number of isolates is called optimal. A graph that admits super vertex sum labeling is called as super vertex sum graph [5].

**Definition 1.** Let $G$ be a graph with $q$ edges. A graph $H$ is called a super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i$, $1 \leq i \leq q$ in such a way that the end vertices of each $e_i$ are identified with the two vertices of the 2-vertex part of $K_{2,m_i}$ after removing the edge $e_i$ from graph $G$. If $m_i$ is varying arbitrarily for each edge $e_i$ then super subdivision is called an arbitrary super subdivision of $G$, [6].

**Definition 2.** Let $K_{1,p}$ and $K_{1,n}$ be two stars. The resulting graph obtained by adding an edge between the center (apex) vertices of $K_{1,p}$ and $K_{1,n}$ is called bi-star, denoted by $B_{p,n}$, [3].

**Definition 3.** A tree is called a spider if it has a center vertex $c$ of degree $k > 1$ and each other vertex is either a leaf (pendent vertex) or has degree 2, [1].

Thus a spider is an amalgamation of $k$ paths with various lengths. If it has $x_1$ paths of length $a_1$, $x_2$ paths of length $a_2$, $\ldots$, $x_n$ paths of length $a_n$, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, \ldots, a_n^{x_n})$, where $a_1 < a_2 < \cdots < a_n$ and $x_1 + x_2 + \cdots + x_n = k$.

**Definition 4.** Let $G_1, G_2, \ldots, G_n$, $n \geq 2$ be $n$ copies of a fixed graph $G$. The graph obtained by adding an edge between $G_i$ and $G_{i+1}$ for $i = 1, 2, \ldots, n-1$ is called the path union of $G$ [7].

2. Optimal Super Vertex Sum Labeling Scheme for Super Subdivision of Bi-star and Path Union of Spiders

Sethuraman and Selvaraju [6], introduced a new method of construction called Super subdivision of graph and proved that arbitrary super subdivision of any
path and cycle are graceful. In this section, we obtain optimal super vertex sum labeling for super subdivision of bi-star and path union of spider.

**Theorem 5.** Super subdivision of bi-star is a super vertex sum graph with $\sigma_{sv}(G) = 2$.

**Proof.** Let $G$ be a bi-star $B_{p,n}$ with central vertices $c_1$ and $c_2$. It has $p+n+2$ vertices and $p+n+1$ edges. By the definition of super subdivision, each edge of $G$ is replaced by complete bipartite graph $K_{2,m}$. The resulting graph $H$ has $(p+n+1)(m+1) + 1$ vertices and $2m(p+n+1)$ edges. Let $u_1, u_2$ be the isolated vertices necessary to super vertex sum label the graph.

Define $f : V(H) \to \{1, 2, \ldots, (p+n+1)(m+1) + 3\}$ as follows.

Choose the vertex with degree $m(p+1)$ as first vertex and label it as 1, i.e., $f(c_1) = 1$. Initialize $i = 1$.

For $i = 1$ to $p$,

- if an unvisited vertex $v_i$ is with $\deg(v_i) = m$ and $d(c_1,v_i) = 2$,
  - then $f(v_i) = i + 1$
- for all unvisited vertices $v_{ij}$ adjacent to $v_i$ and $\deg(v_{ij}) = 2$,
  - then $f(v_{ij}) = i + 1 + j + (m+1)(q-i), \ j = 1 \text{ to } m$
- if an unvisited vertex $v_i$ is with $\deg(v_i) = m(n+1)$ and $d(c_1,v_i) = 2$, then $f(v_i) = p + 2$

for all unvisited vertices $v_{ij}$ with $d(c_1,v_{ij}) = 1$ and $d(c_2,v_{ij}) = 1$ and $\deg(v_{ij}) = 2$, then $f(v_{ij}) = f(c_2) + j + (m+1)(q-p-1), \ j = 1 \text{ to } m$

For $i = (p+2) \text{ to } (p+n+1)$,

- if an unvisited vertex $v_i$ is with $\deg(v_i) = m$ and $d(c_2,v_i) = 2$,
  - then $f(v_{i-1}) = i + 1$
- for all unvisited vertices $v_{ij}$ adjacent to $v_{i-1}$ and $\deg(v_{ij}) = 2$,
  - then $f(v_{(i-1)j}) = i + 1 + j + (m+1)(q-i), \ j = 1 \text{ to } m$

The isolates $u_1$ and $u_2$ are labelled as $(p+n+1)(m+1) + 2$ and $(p+n+1)(m+1) + 3$ respectively.

Hence, super subdivision of bi-star is super vertex sum graph with $\sigma_{sv}(G) = 2$. \hfill $\Box$

**Theorem 6.** Super subdivision of path union of spiders is a super vertex sum graph with $\sigma_{sv}(G) = 2$.

**Proof.** Consider a spider with $k$ paths. Let $l_i$ be the be length of path $P_i$,....
1 ≤ i ≤ k. The graph $G(p, q)$, path union of spiders, is obtained by taking $n$ copies of the spider and adding an edge between center vertex (vertex with maximum degree) of each copy of the spider. Therefore, the number of vertices $p = n(l_1 + l_2 + \ldots + l_k + 1)$ and the number of edges $q = n(l_1 + l_2 + \ldots + l_k) + (n - 1)$. Super subdivide the graph $G$ by replacing each edge by $K_{2,m}$. The resulting graph $H$ has $p + mq$ vertices and $2mq$ edges.

Figure 1: Super subdivided Path Union of Spider

Define $f : V(H) \to \{1, 2, \ldots, p + mq + 2\}$ as follows.

Choose the vertex with degree $m(k + 1)$ as the center vertex or apex vertex of the first copy of spider and label it as $f(c_1) = 1$. All the remaining vertices are named as follows. Let $t = 1$.

for $g = 1$ to $n$
\begin{align*}
i &= 1, \\
&\text{for } x = 1 \text{ to } k \\
&\text{for } l = 1 \text{ to } l_x \\
&\text{if } l = 1, \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with degree } m \text{ or } 2m \text{ and } d(c_g, v) = 2 \\
f(v_{gi}) &= t + 1 \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with } d(c_g, v) = 1 \text{ and } d(v_{gi}, v) = 1 \text{ and degree } 2 \\
f(v_{gij}) &= f(v_{gi}) + j + (m + 1)(q - t); \ j = 1 \text{ to } m \\
i &= i + 1; \ t = t + 1; \ l = l + 1 \\
&\text{if } l \neq 1 \text{ and } l \leq l_x \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with degree } m \text{ or } 2m \text{ and } d(c_{g(i-1)}, v) = 2 \\
f(v_{gi}) &= t + 1 \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with } d(c_{g(i-1)}, v) = 1 \text{ and } d(v_{gi}, v) = 1 \text{ and degree } 2 \\
f(v_{gij}) &= f(v_{gi}) + j + (m + 1)(q - t); \ j = 1 \text{ to } m \\
i &= i + 1; \ t = t + 1; \ l = l + 1 \\
x &= x + 1 \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with } d(c_g, v) = 2 \text{ and degree } m(k + 2) \text{ or } m(k + 1) \\
f(c_g+1) &= t + 1 \\
&\text{if } g \leq (n - 1) \\
&\text{for an unlabelled (unvisited) vertex } v \\
&\text{with } d(c_g, v) = 1 \text{ and } d(c_{g+1}, v) = 1 \text{ degree } 2 \\
f(u_{gij}) &= f(c_{g+1}) + j + (m + 1)(q - t); \ j = 1 \text{ to } m \\
t &= t + 1; \ g = g + 1
\end{align*}

The isolates are labeled as \( f(u_i) = p + mq + i; \ i = 1 \text{ to } 2. \)

Hence, super subdivision of path union of spider is super vertex sum graph with \( \sigma_{sv}(G) = 2. \)

\(\square\)

**Example 1.** Consider a spider with 4 paths of length \( l_1 = 3, \ l_2 = 2, \)
For path union of spiders $G$, we take $n = 3$ copies (say), then $G$ will have $p = 30$ vertices and $q = 29$ edges. Let $G$ be super subdivided by $K_{2,m}$ where $m = 3$. Therefore, the resulting graph has $p + mq = 30 + 3 \times 29 = 117$ vertices and $2mq = 2 \times 3 \times 29 = 174$ edges.

The super vertex sum labeling for the super subdivided path union of spider is given in Figure 1.

3. Algorithm to Construct Super Vertex Sum Labeling of Super Subdivision of Caterpillar

Let $G$ be a Caterpillar with $p$ vertices and $q$ edges. Let $c$ be the number of internal vertices of the caterpillar. Super subdivide the graph $G$ with $K_{2,m}$. The resulting graph $H$ has $p + mq$ vertices and $2mq$ edges. Let $u_1$ and $u_2$ be the two isolates necessary for the graph $H$. Choose the first vertex of $H$ as the vertex with degree $m$ and which has a vertex of degree greater than $m$ at distance $2c$ (i.e., the head or tail of the caterpillar). We follow DFS algorithm with a modification: whenever we find a new vertex $v$ having degree greater than $m$, we first visit all the vertices at distance 2 and of degree $m$ if any, before proceeding to next vertex with degree greater than $m$.

Define $f : V(H) \rightarrow \{1, 2, \ldots, p + mq + 2\}$ as follows:

- Step 1: label the first vertex chosen, $v_1$ as 1.
- Step 2: assign $i = 1$, $l = 1$
- Step 3: for any unvisited vertex $v$ with $d(v_i, v) = 1$ and $deg(v) = 2$
  \[
  \begin{cases} 
  \text{for } (j = 1 \text{ to } m) \\
  f(v_{ij}) = i + 1 + j + (m + 1)(q - i)
  \end{cases}
  \]
- Step 4: for the vertex $v$ with $d(v_i, v) = 1$ and $deg(v) > m$
  \[
  f(v_i) = i + 1
  \]
  increment $i$
- Step 5: if ($l \leq c$) go to Step 6; else go to Step 8.
- Step 6: for an unvisited vertex (if any) with $d(v_i, v) = 2$ and $deg(v) = m$
  \[f(v_{i+1}) = i + 1\]
  go to Step 7
- Step 7: for an unvisited vertex (if any) with $d(v_i, v) = 2$
and \( \text{deg}(v) > m \)
\[
\begin{aligned}
\text{for } (j = 1 \text{ to } m) \\
f(v_{ij}) &= i + 1 + j + (m + 1)(q - i) \\
l &= l + 1 \\
f(c_i) &= i + 1 \\
i &= i + 1 \\
go \text{ to Step 5}
\end{aligned}
\]

• Step 7: for all unvisited vertex \( v \) with \( d(v_{i+1}, v) = 1 \) and \( \text{deg}(v) = 2 \)
\[
\begin{aligned}
\text{for } (j = 1 \text{ to } m) \\
f(v_{ij}) &= i + 1 + j + (m + 1)(q - i) \\
i &= i + 1 \\
\text{if } (i = p) \text{ go to Step 8; else go to Step 6}
\end{aligned}
\]

• Step 8: \( f(u_1) = p + mq + 1 \) and \( f(u_2) = p + mq + 2 \)
• Step 9: stop

**Example 2.** Consider a caterpillar with \( p = 15 \) vertices and \( q = 14 \) edges. The number of internal vertices \( c = 3 \). Let \( G \) be super subdivided by \( K_{2,m} \) where \( m = 3 \). Therefore, the resulting graph has \( p + mq = 15 + 3 \times 14 = 57 \) vertices and \( 2mq = 2 \times 3 \times 14 = 84 \) edges.

The super vertex sum labeling for the super subdivided caterpillar is given in Figure 2.

### 4. Conclusion

In this paper, we provided optimal super vertex sum labeling for the super subdivision of bi-star and path union of spiders. Further we provided the algorithm to construct super vertex sum labeling of super subdivision of caterpillar. For further studies, optimal super vertex sum labeling for other family of trees, the optimal super vertex sum number for arbitrary super subdivision of the same graphs can be explored.
Figure 2: Super Vertex Sum Labeling for the Super Subdivided Caterpillar

References


