International Journal of Applied Mathematics

Volume 31 No. 4 2018, 641-648

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v31i4.9

SOME GRAPHS WITH SUPER VERTEX SUM NUMBER 2

Gerard Rozario Joseph[1], Joseph Varghese Kureethara[3 §]

[1] Bharathiar University
Coimbatore, 641046, INDIA

[1]IFET College of Engineering
Villupuram, 605108, INDIA

[3]CHRIST (Deemed to be University)
Department of Mathematics and Statistics
Bangalore, 560029, INDIA

Abstract: In this paper we give optimal super vertex sum labeling scheme for super subdivision of bi-star, path union of spider and algorithm to construct super vertex sum labeling of super subdivision of Caterpillar.

AMS Subject Classification: 05C78, 05C99, 11Z05
Key Words: sum labeling, sum number, sum graph, isolates

1. Introduction

We considered only simple, finite and undirected graphs for the study. For all terminologies and notations which are not elucidated in detail, we refer [3] and [2]. The sum labeling of graphs was introduced by Harary [4]. Let $G(p, q)$ be a graph and $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijective mapping. For every pair of adjacent vertices $u, v \in V(G)$ let $f(u) + f(v) = f(w)$ for some vertex $w \in V(G)$. Let $\mu_f(G)$ be the maximum of $f(u)$ for vertex $\forall u \in V(G)$. If $\mu_f(G) = |V(G)|$, then f is called *super vertex sum labeling*, [5].

A graph that admits super vertex sum labeling will be always disconnected. This is because we need isolated vertices to super vertex sum label a graph. The least number of isolates needed to super vertex sum label the graph G is called

Received: May 3, 2018

§Correspondence author

super vertex sum number of the graph, denoted by $\sigma_{sv}(G)$. It is proved that the lower bound of $\sigma_{sv}(G)$ is $\delta(G)$, [5]. It is also shown that super subdivision of path, cycles, stars and spiders are super vertex sum graph with $\sigma_{sv}(G) = 2$ [5].

The labeling with minimum number of isolates is called *optimal*. A graph that admits *super vertex sum labeling* is called as *super vertex sum graph* [5].

Definition 1. Let G be a graph with q edges. A graph H is called a *super subdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of the 2-vertex part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then super subdivision is called an *arbitrary super subdivision* of G, [6].

Definition 2. Let $K_{1,p}$ and $K_{1,n}$ be two stars. The resulting graph obtained by adding an edge between the center (apex) vertices of $K_{1,p}$ and $K_{1,n}$ is called bi-star, denoted by $B_{p,n}$, [3].

Definition 3. A tree is called a spider if it has a center vertex c of degree $k > 1$ and each other vertex is either a leaf (pendent vertex) or has degree 2, [1].

Thus a spider is an amalgamation of k paths with various lengths. If it has x_1 paths of length a_1, x_2 paths of length a_2, $\dots$, x_n paths of length a_n, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, \dots, a_n^{x_n})$, where $a_1 < a_2 < \cdots < a_n$ and $x_1 + x_2 + \cdots + x_n = k$.

Definition 4. Let $G_1, G_2, \dots G_n$, $n \geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1} for $i = 1, 2, \dots, n-1$ is called the *path union* of G [7].

2. Optimal Super Vertex Sum Labeling Scheme for Super Subdivision of Bi-star and Path Union of Spiders

Sethuraman and Selvaraju [6], introduced a new method of construction called Super subdivision of graph and proved that arbitrary super subdivision of any path and cycle are graceful. In this section, we obtain optimal super vertex sum labeling for super subdivision of bi-star and path union of spider.

Theorem 5. *Super subdivision of bi-star is a super vertex sum graph with* $\sigma_{sv}(G) = 2$.

Proof. Let G be a bi-star $B_{p,n}$ with central vertices c_1 and c_2. It has $p+n+2$ vertices and $p+n+1$ edges. By the definition of super subdivision, each edge of G is replaced by complete bipartite graph $K_{2,m}$. The resulting graph H has $(p+n+1)(m+1)+1$ vertices and $2m(p+n+1)$ edges. Let u_1, u_2 be the isolated vertices necessary to super vertex sum label the graph.

Define $f : V(H) \to \{1, 2, \ldots, (p+n+1)(m+1)+3\}$ as follows.

Choose the vertex with degree $m(p+1)$ as first vertex and label it as 1, i.e., $f(c_1) = 1$. Initialize $i = 1$.

For $i = 1$ *to* p,

$$\begin{cases} \text{if an unvisited vertex } v_i \text{ is with } \deg(v_i)\text{=m and } d(c_1, v_i)\text{=2,} \\ \text{then } f(v_i) = i+1 \\ \text{for all unvisited vertices } v_{ij} \text{ adjacent to } v_i \text{ and } \deg(v_{ij})\text{=2,} \\ \text{then } f(v_{ij}) = i+1+j+(m+1)(q-i),\ j = 1 \textit{ to } m \end{cases}$$

if an unvisited vertex v_i is with deg(v_i)=m(n+1) and $d(c_1, v_i)$=2, then $f(c_2) = p+2$.

for all unvisited vertices v_{ij} with $d(c_1, v_{ij}) = 1$ and $d(c_2, v_{ij}) = 1$ and deg(v_{ij})=2, then $f(v_{ij}) = f(c_2) + j + (m+1)(q-p-1)$, $j = 1$ *to* m.

For $i = (p+2)$ *to* $(p+n+1)$,

$$\begin{cases} \text{if an unvisited vertex } v_i \text{ is with } \deg(v_i)\text{=m and } d(c_2, v_i)\text{=2,} \\ \text{then } f(v_{i-1}) = i+1 \\ \text{for all unvisited vertices } v_{ij} \text{ adjacent to } v_{i-1} \text{ and } \deg(v_{ij})\text{=2,} \\ \text{then } f(v_{(i-1)j}) = i+1+j+(m+1)(q-i),\ j = 1 \textit{ to } m \end{cases}$$

The isolates u_1 and u_2 are labelled as $(p+n+1)(m+1)+2$ and $(p+n+1)(m+1)+3$ respectively.

Hence, super subdivision of bi-star is super vertex sum graph with $\sigma_{sv}(G) = 2$. □

Theorem 6. *Super subdivision of path union of spiders is a super vertex sum graph with* $\sigma_{sv}(G) = 2$.

Proof. Consider a spider with k paths. Let l_i be the be length of path P_i,

$1 \leq i \leq k$. The graph $G(p, q)$, path union of spiders, is obtained by taking n copies of the spider and adding an edge between center vertex (vertex with maximum degree) of each copy of the spider. Therefore, the number of vertices $p = n(l_i + l_2 + \cdots + l_k + 1)$ and the number of edges $q = n(l_i + l_2 + \cdots + l_k) + (n-1)$. Super subdivide the graph G by replacing each edge by $K_{2,m}$. The resulting graph H has $p + mq$ vertices and $2mq$ edges.

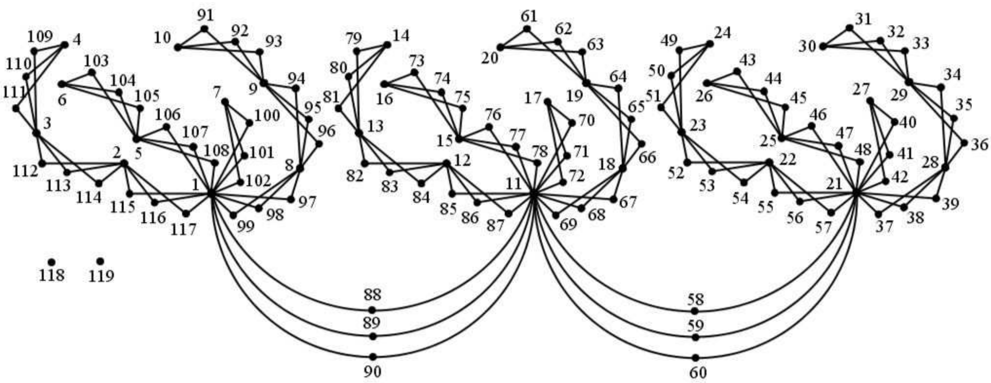

Figure 1: Super subdivided Path Union of Spider

Define $f : V(H) \to \{1, 2, \ldots, p + mq + 2\}$ as follows.

Choose the vertex with degree $m(k+1)$ as the center vertex or apex vertex of the first copy of spider and label it as$f(c_1) = 1$. All the remaining vertices are named as follows. Let $t = 1$.

for $g = 1$ *to* n

$$\begin{cases} i = 1, \\ for\ x = 1\ to\ k \\ \begin{cases} for\ l = 1\ to\ l_x \\ \begin{cases} if\ l = 1, \\ \begin{cases} for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ degree\ m\ or\ 2m\ and\ d(c_g, v) = 2 \\ f(v_{gi}) = t + 1 \\ for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ d(c_g, v) = 1\ and\ d(v_{g_i}, v) = 1\ and\ degree\ 2 \\ f(v_{g_{ij}}) = f(v_{gi}) + j + (m+1)(q-t);\ j = 1\ to\ m \\ i = i + 1;\ t = t + 1;\ l = l + 1 \end{cases} \\ if\ l \neq 1\ and\ l \leq l_x \\ \begin{cases} for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ degree\ m\ or\ 2m\ and\ d(c_{g(i-1)}, v) = 2 \\ f(v_{gi}) = t + 1 \\ for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ d(c_{g(i-1)}, v) = 1\ and\ d(v_{g_i}, v) = 1\ and\ degree\ 2 \\ f(v_{g_{ij}}) = f(v_{gi}) + j + (m+1)(q-t);\ j = 1\ to\ m \\ i = i + 1;\ t = t + 1;\ l = l + 1 \end{cases} \end{cases} \\ x = x + 1 \end{cases} \\ for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ d(c_g, v) = 2\ and\ degree\ m(k+2)\ or\ m(k+1) \\ f(c_{g+1}) = t + 1 \\ if\ g \leq (n-1) \\ \begin{cases} for\ an\ unlabelled(unvisited)\ vertex\ v \\ with\ d(c_g, v) = 1\ and\ d(c_{g+1}, v) = 1\ degree\ 2 \\ f(u_{g_{ij}}) = f(c_{g+1}) + j + (m+1)(q-t);\ j = 1\ to\ m \end{cases} \\ t = t + 1;\ g = g + 1 \end{cases}$$

The isolates are labeled as $f(u_i) = p + mq + i;\ i = 1\ to\ 2$.

Hence, super subdivision of path union of spider is super vertex sum graph with $\sigma_{sv}(G) = 2$. □

Example 1. Consider a spider with 4 paths of length $l_1 = 3$, $l_2 = 2$,

$l_3 = 1$, $l_4 = 3$. For path union of spiders G, we take $n = 3$ copies (say), then G will have $p = 30$ vertices and $q = 29$ edges. Let G be super subdivided by $K_{2,m}$ where $m = 3$. Therefore, the resulting graph has $p + mq = 30 + 3\text{x}29 = 117$ vertices and $2mq = 2\text{x}3\text{x}29 = 174$ edges.

The super vertex sum labeling for the super subdivided path union of spider is given in Figure 1.

3. Algorithm to Construct Super Vertex Sum Labeling of Super Subdivision of Caterpillar

Let G be a Caterpillar with p vertices and q edges. Let c be the number of internal vertices of the caterpillar. Super subdivide the graph G with $K_{2,m}$. The resulting graph H has $p + mq$ vertices and $2mq$ edges. Let u_1 and u_2 be the two isolates necessary for the graph H. Choose the first vertex of H as the vertex with degree m and which has a vertex of degree greater than m at distance $2c$ (i.e., the head or tail of the caterpillar). We follow DFS algorithm with a modification: whenever we find a new vertex v having degree greater than m, we first visit all the vertices at distance 2 and of degree m if any, before proceeding to next vertex with degree greater than m.

Define $f : V(H) \to \{1, 2, \ldots, p + mq + 2\}$ as follows:

- Step 1: label the first vertex chosen, v_1 as 1.
- Step 2: assign $i = 1$, $l = 1$
- Step 3: for any unvisited vertex v with $d(v_i, v) = 1$ and $deg(v) = 2$
  $\begin{cases} for\ (j = 1\ to\ m) \\ f(v_{ij}) = i + 1 + j + (m + 1)(q - i) \end{cases}$
- Step 4: for the vertex v with $d(v_i, v) = 1$ and $deg(v) > m$
  $f(c_i) = i + 1$
  increment i
- Step 5: if $(l \leq c)$ go to Step 6; else go to Step 8.
- Step 6: for an unvisited vertex (if any) with $d(c_l, v) = 2$ and $deg(v) = m$
  $do \begin{cases} f(v_{i+1}) = i + 1 \\ go\ to\ Step\ 7 \end{cases}$ else for an unvisited vertex (if any) with $d(c_l, v) = 2$

and $deg(v) > m$

$$\begin{cases} for\ (j = 1\ to\ m) \\ f(v_{ij}) = i + 1 + j + (m+1)(q-i) \\ l = l + 1 \\ f(c_l) = i + 1 \\ i = i + 1 \\ go\ to\ Step\ 5 \end{cases}$$

- Step 7: for all unvisited vertex v with $d(v_{i+1}, v) = 1$ and $deg(v) = 2$

$$\begin{cases} for\ (j = 1\ to\ m) \\ f(v_{ij}) = i + 1 + j + (m+1)(q-i) \\ i = i + 1 \\ if\ \ (i = p)\ go\ to\ Step\ 8;\ else\ go\ to\ Step\ 6 \end{cases}$$

- Step 8:$f(u_1) = p + mq + 1$ and $f(u_2) = p + mq + 2$
- Step 9: stop

Example 2. Consider a caterpillar with $p = 15$ vertices and $q = 14$ edges. The number of internal vertices $c = 3$. Let G be super subdivided by $K_{2,m}$ where $m = 3$. Therefore, the resulting graph has $p + mq = 15 + 3x14 = 57$ vertices and $2mq = 2x3x14 = 84$ edges.

The super vertex sum labeling for the super subdivided caterpillar is given in Figure 2.

4. Conclusion

In this paper, we provided optimal super vertex sum labeling for the super subdivision of bi-star and path union of spiders. Further we provided the algorithm to construct super vertex sum labeling of super subdivision of caterpillar. For further studies, optimal super vertex sum labeling for other family of trees, the optimal super vertex sum number for arbitrary super subdivision of the same graphs can be explored.

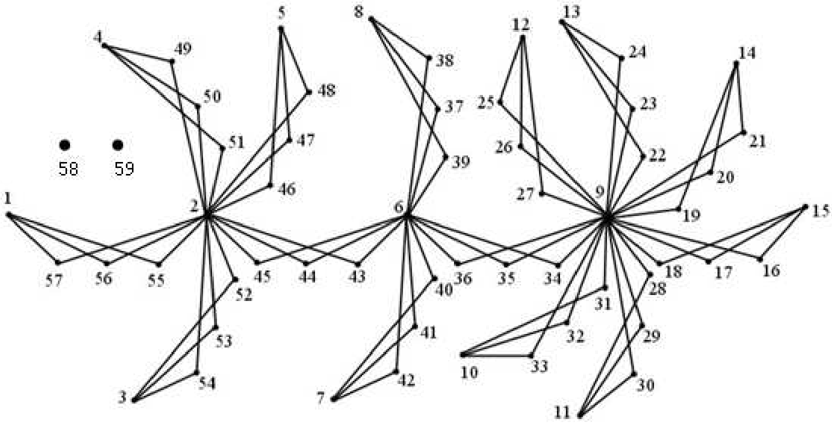

Figure 2: Super Vertex Sum Labeling for the Super Subdivided Caterpillar

References

[1] P.T. Chung and S.M. Lee, On the super edge-graceful spiders of even orders, *J. Combinatorial Math. Combinatorial Computing*, **64** (2008), 3-17.

[2] J.A. Gallian, A dynamic survey of graph labeling, DS6 *The Electronics J. Combinatorics*, Version 20 (2017).

[3] F. Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts (1969).

[4] F. Harary, Sum graphs and Difference graphs, *Congressus Numerantium*, **72** (1990), 101-108.

[5] G.R. Joseph and J.V. Kureethara, Super vertex sum labeling of graphs, *Int. J. Pure Appl. Math.*, **118**, No 10 (2018), 293-300; DOI: 10.12732/ijpam.v118i10.70.

[6] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary super subdivisions of graphs, *Indian J. Pure Appl. Math.*, **32**, No 7 (2001), 1059-1064.

[7] S.K. Vaidya, N.A. Dani, K.K. Kanani and P.L. Vihol, Cordial and 3-equitable labeling for some star related graphs, *Int. Mathematical Forum*, **4**, No 31 (2009), 1543-1553.