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# PENDANT NUMBER OF GRAPHS

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**Abstract:** A decomposition of a graph $G$ is a collection of its edge disjoint sub-graphs such that their union is $G$. A path decomposition of a graph is a decomposition of it into paths. In this paper, we define the pendant number $\Pi_p$ as the minimum number of end vertices of paths in a path decomposition of $G$ and determine this parameter for certain fundamental graph classes.

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**Key Words:** decomposition, path decomposition, Hamilton decomposition, pendant number

## 1. Introduction

A *decomposition* of a graph $G$ is a collection of subgraphs $\mathcal{H} = \{H_1, H_2, \dots H_k; 1 \leq i \leq k\}$ of $G$ such that $\{E(H_1), E(H_2), \dots E(H_k)\}$ is a partition of $E(G)$. Let $P_n$ be a path of length $n$. The vertices of $P_n$ with degree one are called its end vertices and all other vertices are called its internal vertices. A *path-decomposition* of a graph $G$ is defined in Heinrich [5] as a partition of its edge-set into subgraphs each of which is a path in $G$.

The *path decomposition number* of a graph $G$, denoted by $\Pi(G)$, is defined as the minimum cardinality of a path decomposition of $G$, [2].

In this paper, we introduce the term pendant number of a graph and discuss this parameter for certain classes of fundamental graphs.

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For terms and definitions in Graph Theory, we refer to [4] and [7], and for graph classes we refer to ISGCI, [8]. Unless mentioned otherwise, all graphs we consider in this paper are undirected, simple, finite and connected.

## 2. Pendant Number

**Definition 1.** The *pendant number* of a graph $G$, denoted by $\Pi_p(G)$, is the least number of vertices in a graph such that they are the end vertices of a path in a given path decomposition of a graph.

If $V_P(G)$ denotes the set of all $u \in V(G)$ such that $u$ is an end vertex of a path in $P$-decomposition in $G$, then $\Pi_p(G) = \min\{|V_p(G)|\}$. First, recall the following theorem on the path decomposition number of a tree.

**Theorem 2.** *For any tree $T$, the path decomposition number $\Pi_T = \frac{l}{2}$, where $l$ is the number of vertices of odd degree (see Stanton [6]).*

Invoking Theorem 2, we begin with a sharp bound for the pendant number $\Pi_p(G)$.

**Theorem 3.** *Let $G$ be a connected graph with $n$ vertices. If $G$ has $l$ odd degree vertices, then $l \leq \Pi_p(G) \leq n$.*

*Proof.* Let $G$ be a graph on $n$ vertices; out of which, $l$ are odd degree vertices.

*Claim:* Every odd degree vertex in $G$ is an end vertex of some paths in the path decomposition $\mathcal{P}$ of $G$.

To prove this claim, let $v$ be an odd degree vertex of $G$ with $d_G(v) = 2r+1$. If possible, let $v$ be not an end vertex of a path in $\mathcal{P}$. Then $v$ must be an internal vertex of every path passing through it. Let $P_{(1)}, P_{(2)}, \ldots, P_{(r)}$ be the paths in $\mathcal{P}$ which pass through $v$.

It is to be noted that there are $r-1$ paths passing through $v$ in $G - P_{(1)}$ and $G_1 = d_{G-P_{(1)}}(v) = d_G(v) - 2$. In a similar manner, we remove the paths one by one. The reduced graphs at each stage and the corresponding degrees of the vertex $v$ can be as given in Table 1.

Then, from the above table, we have $d_{G_r}(v) = d_G(v) - 2(r-1) - 2 = 1$. Therefore, $v$ is not an internal vertex in $G_r$ and $v$ is an end vertex of $G_r$, a contradiction. Hence, $v$ is an end vertex of the path $P_{(r)}$ in $\mathcal{P}$. Now we proceed to establish the bounds for $\Pi_p(G)$. That is, each odd degree vertex in $G$ will

| Graph | $d(v)$ |
|---|---|
| $G_1 = G - (P_{(1)})$ | $d_G(v) - 2$ |
| $G_2 = G - (P_{(1)} \cup P_{(2)})$ | $d_G(v) - 4$ |
| $G_3 = G - (P_{(1)} \cup P_{(2)} \cup P_{(3)})$ | $d_G(v) - 6$ |
| ...... | ...... |
| $G_{r-1} = G - (P_{(1)} \cup P_{(2)} \cup \ldots \cup P_{(r-1)})$ | $d_G(v) - 2(r-1)$ |

Table 1

be an end vertex of some paths in $\mathcal{P}$. That is,

$$l \leq \Pi_p(G). \tag{1}$$

Again, the maximum number of vertices in the given graph $G$ be $n$. Suppose all these vertices are counted as the end vertices of different path decomposition in $G$, then

$$\Pi_p(G) \leq n. \tag{2}$$

From (1) and (2)we get, $l \leq \Pi_p(G) \leq n$. □

**Corollary 4.** *If $G$ is a connected graph with $n \geq 2$ vertices, then $2 \leq \Pi_p(G) \leq n$.*

For example, the lower bound is attained if $G$ is a path $P_n$ on $n \geq 2$ and the upper bound is attained if $G$ is a star $K_{1,n}$ with odd $n$.

## 3. Pendant Number of Acyclic Graphs

An acyclic connected graph is called tree. Since, trees are one of the most basic classes of graphs, we start with the basic properties of the pendant number of trees.

Since every path $P_n$ has exactly two pendent vertices, we have $\Pi_p(P_n) = 2$. The following result is immediate from the fact that every pendent vertex of a tree is an end vertex of some path in $T$.

**Proposition 5.** *Let $T$ be a tree with $p$ number of pendent vertices, then $p \leq \Pi_p(G)$.*

**Theorem 6.** *Let $T$ be a tree with $n \geq 2$ vertices. Then $\Pi_p(T) = 2$ if and only if $T$ is a path $P_n$.*

*Proof.* It can be easily verify that if $T$ is the path $P_n$, then $\Pi_p(P_n) = 2$. Conversely assume that $\Pi_p(T) = 2$ and the tree $T$ is not a path. Then, $T$ contains a vertex of degree at least three. Let $v$ be a vertex in $T$ with degree at least three in $T$. Then, there is at least two edge disjoint paths incident on the vertex $v$. Therefore, in counting the number of vertices in $\Pi_p(T)$, we need at least four vertices (see Figure 1), a contradiction. □

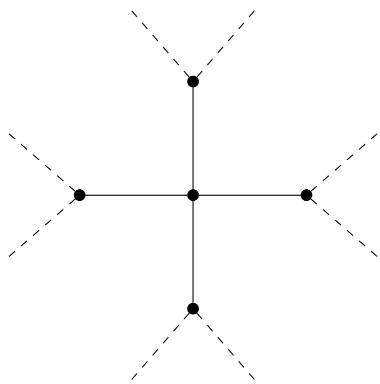

Figure 1

For a binary tree $T$, the root vertex is the only even degree vertex. By claim 1 of Theorem 3, we see that the pendant number $\Pi_p(T)$ is always $n - 1$.

**Corollary 7.** *If $G$ is the star graph $K_{1,n}$ on $n \geq 2$ vertices, then*

$$\Pi_p(G) = \begin{cases} n & \text{if } n \text{ is even;} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

*Proof. Case-1.* Let $n$ be even. Let $v_1, v_2, \ldots v_n$ be the pendant vertices of $K_{1,n}$. Now, every path in the minimal path decomposition is $P_3$ and of the form $v_{2i-1}vv_{2i}; i = 1, 2, \ldots \frac{n}{2}$, where $v$ is the vertex with maximum degree. Therefore, all pendant vertices of $K_{1,n}$ will be an end vertex of a path in $\mathcal{P}$. Therefore,

$$\Pi_p(K_{1,n}) = n. \tag{3}$$

*Case-2.* Let $n$ be odd. Then, as explained in the above case, all paths except one has the form $v_{2i-1}vv_{2i}; 1 \leq i \leq \frac{n-1}{2}$ and the last on in $\mathcal{P}$ is a path

$vv_n$. That is, all vertices in $K_{1,n}$ are in $V_p(G)$. Therefore,

$$\Pi_p(K_{1,n}) = n + 1. \tag{4}$$

From (3) and (4)we get,

$$\Pi_p(G) = \begin{cases} n, & \text{if } n \text{ is even;} \\ n+1, & \text{if } n \text{ is odd.} \end{cases}$$

□

**Theorem 8.** *Let $T$ be tree on $n$ vertices, of which $k$ vertices are of even degree. Then, $\Pi_p(T) = n - k$.*

*Proof.* Given that $T$ has $k$ even degree vertices. Note that all these $k$ vertices are internal vertices of $T$ and will be an internal vertex of any maximal paths in $T$. In other words, these $k$ vertices will not be the end vertices of any paths in the optimal path decomposition of $T$. We also note that all the remaining $n-k$ vertices are of odd degree and by the claim of Theorem 3, these $n-k$ vertices will be the end vertices of some paths in an optimal path decomposition of $T$. Therefore, $\Pi_p T = n - k$. □

## 4. Pendant Number of Cyclic Graphs

Now, we move on to the properties of cyclic graphs.

**Proposition 9.** *If $G$ is the cycle $C_n$ on $n \geq 3$ vertices, then $\Pi_p(G) = 2$.*

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of $C_n$. Let us decompose $v_1 - v_2 - \ldots - v_n$ as the path $P_1$ with length $n-1$ having end vertices $v_1$ and $v_n$ and the edge $e = v_1 v_n$ as the another path $P_2$ with length one. Sine $P_1$ and $P_2$ have the same end vertices $v_1$ and $v_n$, $V_{p_i}(G) = \{v_1, v_n\}$. Therefore, $\Pi_p(G) = 2$. □

**Theorem 10.** *For a unicyclic graph $G$ of order $n$; $n \geq 3$ with $l$ odd degree vertices, we have*

$$\Pi_p(G) = \begin{cases} 2 & \text{if } m = 0; \\ l+1 & \text{if } m = 1; \\ l & \text{otherwise.} \end{cases}$$

*where $m$ is the number of vertices on $C$ with $deg(v) \geq 2$.*

*Proof.* *Case 1:* Let $m = 0$. In this case, $G$ itself is a cycle and hence the proof is immediate.

*Case 2:* Let $m = 1$. Let $x$ be the vertex in $C$ such that $deg(x) > 2$. Choose a vertex $y \neq x$ in $C$. Let $u, v \in C$ such that $yux, yvx$ are partitions of the cycle $C$.

Here, we need to consider two sub cases as follows:

*Subcase 2.1:* Let $deg(x)$ be even. Then, definitely all the $l$ odd degree vertices are in $G - C$. Without loss of generality, $yux$ and $yvx$ can be extended to any two odd degree vertices in $G - C$ and hence $y$ alone will be the end vertex in $C$. Thus, $\Pi_p(G) = l + |y| = l + 1$.

*Subcase 2.2:* Let $deg(x)$ be odd. We can decompose $G$ into two trees $T_1$ and $T_2$ in such a way that $y$ is the common end vertex for one path in $T_1$ and another path in $T_2$. (see Figure 2 for illustration). Let $T_1 = yux$ and $T_2 = G - yux$. Now the remaining $l - 1$ odd degree vertices of $G$ are in $G - C = T_2 - yvx$. Since $T_1$ and $T_2$ are trees, by Theorem 3, $\Pi_p(T_1) = 2$ and $\Pi_p(T_2) = l$. But the vertex $y$ is counted twice; once in $T_1$ and again in $T_2$. Therefore, $\Pi_p(G) = \Pi_p(T_1) + \Pi_p(T_2) - 1 = 2 + l - 1 = l + 1$.

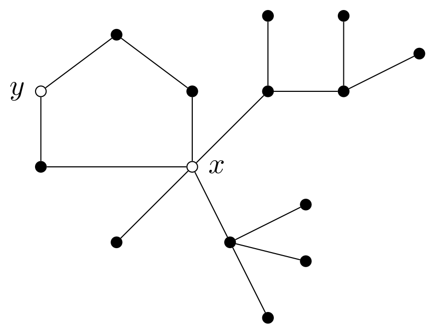


Figure 2

*Case 3:* Let $m > 1$. Let $\Pi_p(G) \neq l$. Since $l$ is the lower bound of $\Pi_p(G)$, it can be noted that the only possible case here is $\Pi_p(G) > l$. If so, there will be at least one even degree vertex $w$ in $G$ such that $w$ is an end vertex. It implies that either $w \in C$ or $w \in G - C$. Here the following two sub cases arise:

*Subcase 3.1:* Let $w \in C$. Without loss of generality, assume m=2. Let $x, y$ be the two odd degree vertices in $C$. Then there will be at least three trees $T_1, T_2$ and $T_3$ such that $yv_iw$ is a path in $T_1$, $wv_jx$ is a path in $T_2$ and $yv_kx$ is a path in $T_3$; for $v_i, v_j, v_k \in C$. Then, we can join the paths $yv_iw$, $wv_jx$ to get a single path. Therefore, $w \notin C$.

*Subcase 3.2:* Let $w \in G - C$. Since $G$ is unicyclic, $G - C$ is a tree. Hence no even degree vertex can be an end point of a path in $G - C$. Therefore, $w \notin G - C$. Hence no even degree vertex $w$ in $G$ such that $w$ is an end vertex of any path in the path decomposition concerned. Therefore $\Pi_p(G) \not> l$. Hence $\Pi_p(G) = l$. □

A regular graph $G$ with edge-set $E(G)$, is said to have a Hamilton decomposition (see Alspach [1]) (or be Hamilton decomposible) if either

(i) $deg(G) = 2d$ and $E(G)$ can be partitioned into $d$ Hamilton cycles.

(ii) $deg(G) = 2d+1$ and $E(G)$ can be partitioned into $d$ Hamilton cycles and a perfect matching.

In order to distinguish the two cases mentioned above, we call the decomposition which satisfy the condition (i) as *Hamilton decomposition of first kind* and the decomposition which satisfy the condition (ii) as *Hamilton decomposition of second kind.*

**Theorem 11.** *If $G$ has a Hamilton decomposition of first kind, then $\Pi_p(G) = 2$.*

*Proof.* Let $G$ be an $r$-regular graph of even degree. A Hamiltonian cycle covers the entire vertex set of $G$ and the number of cycles will be $\frac{r}{2}$. Since $\Pi_p(C_n) = 2$ and each $C_n$ is running on the same vertices, we have, $\Pi_p(G) = 2$. □

**Proposition 12.** *For a complete graph $K_n$ on $n$ vertices, where $n$ is odd, there will be $\frac{n-1}{2}$ edge-disjoint cycles of length $n$.*

If $n$ is even, we note that all vertices of the complete graph $K_n$ are odd degree vertices and hence by Theorem 3, the pendant number is $n$. The following theorem discusses the pendant number of a complete graph of odd order.

**Theorem 13.** *For $n \geq 3$, $\Pi_p(K_n) = 2$ if and only if $n$ is odd.*

*Proof.* When $n$ is odd, $K_n$ is an $(n-1)$-regular graph. Hence, it is clearly a graph with Hamilton decomposition of first kind. Hence, by Theorem 11, $\Pi_p(K_n) = 2$.

Conversely, take a complete graph $K_n$ on $n \geq 3$ vertices such that $\Pi_p(G) = 2$. Assume, if possible, that $n$ is even. Then, all its vertices are of odd degree $n-1$, a contradiction. Hence, $n$ cannot be even, completing the proof. □

The above facts lead to an interesting result as given below:

**Theorem 14.** *For a connected graph $G$ of order $n$, we have $2 \leq \Pi_p(G) \leq n$. The lower bound is attained if $G$ is a Hamilton decomposition of first kind and the upper bound is attained if $G$ is a Hamilton decomposition of second kind.*

Having the pendant number of acyclic graphs and cyclic graphs in hand, we shall move on to find the pendant number of complete bipartite graphs.

**Theorem 15.** *For a complete bipartite graph $K_{m,n}$ with $m \leq n$,*

$$\Pi_p(K_{m,n}) = \begin{cases} m+n, & \text{if both } m, n \text{ are odd;} \\ m, & \text{if } m \text{ is even and } n \text{ is odd;} \\ n, & \text{if } m \text{ is odd and } n \text{ is even;} \\ 2, & \text{if both } m \text{ and } n \text{ are even.} \end{cases}$$

*Proof.* Let $U, V$ be the partition of the vertex set of $K_{m,n}$ with $|U| = m$ and $|V| = n$.

*Case 1*: Let $m, n$ be odd. Hence, being all vertices of the graph concerned are of odd degree, by Theorem 3, $\Pi_p(K_{m,n}) \geq m+n$. Since, $|V(K_{m,n})| = m+n$, $\Pi_p(K_{m,n}) \leq m+n$. Hence, $\Pi_p(K_{m,n}) = m+n$.

*Case 2*: Let $m$ be even and $n$ be odd. Here, $deg(u_i) = n; u_i \in U$. Hence, by Theorem 3, $m \leq \Pi_p(K_{m,n})$.

The collection $\mathcal{P}$ of edge disjoint 2-paths of $K_{m,n}$, defined by $\mathcal{P} = \{u_j v_i u_k : u_j, u_k \in U, v_i \in V, 1 \leq j \neq k \leq m, 1 \leq i \leq n\}$ clearly forms a path decomposition of the graph $K_{m,n}$. Therefore $\Pi_p(K_{m,n}) \leq m$, Thus, $\Pi_p(K_{m,n}) = m$.

*Case 3*: Let $m$ be odd and $n$ be even. The proof follows exactly as mentioned in Case-2, by interchanging $m$ and $n$.

*Case 4*: Let both $m$ and $n$ be even. All cycles of length $2m$ passes through every vertex $u_i; i \in U$. Since $K_{m,n}$ has $mn$ edges, the total number of cycles passing through all vertices of $U$ is $\frac{n}{2}$. Since, each cycle contains every vertex $u_i; i \in U$, by Proposition 9, $\Pi_p(K_{m,n}) = 2$. □

## 5. Conclusion

In this paper, we have introduced a new notion namely the pendant number of graphs and determined this parameter for certain fundamental graph classes. We have also proposed bounds for this parameter for arbitrary graphs. Investigating the pendant number of several other graph classes remains open. Comparison between the pendant number and certain other graph parameters such as domination number, graph diameter etc. is also promising. Determining the pendant number of certain derived graphs such as complement, line graphs, total graphs etc. also seem to be promising for future investigations. All these facts highlight a wide scope for further studies in this area.

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