OSCILLATION OF THE EVEN-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract: The oscillation criteria are investigated for all solutions of even-order neutral differential equations. The obtained results are based on the new comparison theorems, that enable us to reduce the problem of the oscillation of the higher order equation to the oscillation of the first order equation. The obtained comparison principles essentially simplify the examination of the studied equations.

AMS Subject Classification: 34K11, 34C10
Key Words: oscillation, even-order, comparison theorem, neutral differential equation

1. Introduction

This work is concerned with oscillation behavior of a class of even-order neutral differential equation

Received: September 12, 2018 © 2018 Academic Publications

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\[
(a(t)[(x(t) + p(t)x(\eta(t)))]^{(n-1)})' + q(t)x^\gamma(\sigma(t)) = 0, \quad t \geq t_0. \quad (E)
\]

Throughout this paper, we use the notation \( z(t) := x(t) + p(t)x(\eta(t)) \) and always assume that the following conditions hold:

(A1) \( n \) even, \( \gamma \) is a ratio of two positive odd integers;

(A2) \( a, p, q \in C([t_0, +\infty)), a, b, q > 0, 0 \leq p(t) \leq p_0 < \infty, \eta(t) \in C^1([t_0, +\infty)), \sigma(t) \in C([t_0, +\infty)), \sigma(t) \leq t, \sigma(t) \) is nondecreasing;

(A3) \( \lim_{t \to +\infty} \eta(t) = \infty, \eta' \geq \eta_0 > 0, \sigma^{-1} \) exists and \( \sigma^{-1} \) is continuously differentiable, where \( \sigma^{-1} \) denotes the inverse function of \( \sigma \).

We also assume that

\[
\int_{t_0}^{\infty} a^{-1/\gamma}(s) \, ds = \infty. \quad (1)
\]

By a solution of (E), we mean a function \( z(t) \in C([T_x, \infty)), T_x \geq t_0, \) which has the property \( a(t)(z^n(t))^\gamma \in C([T_x, \infty)) \) and \( x(t) \) satisfies (E) on \([T_x, \infty)\). We consider only those solutions \( x(t) \) of (E) which satisfy \( \sup\{|x(t)| : t \geq T\} > 0 \) for all \( T \geq T_x \). We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on \([T_x, \infty)\) and otherwise, it is said to be nonoscillatory. Equation (E) is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

Neutral delay equations appear in modeling of networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar; see the Euler equation in some variational problems, in the theory of automatic control and in neuromechanical systems in which inertia plays an important role. We refer the reader to [1-17], and references cited therein. Recently, Zafer [6], Meng et al. [7], B. Baculíková [8, 9, 10], Li et al. [11], C. Zhang et al. [16] and Q. Zhang et al. [17] all are studied the oscillation behavior of even-order neutral differential equations.

The purpose of this paper we extend main results of J. Džurina and B. Baculíková [8], we establish some new criteria in which we relax condition of \( \eta \circ \sigma = \sigma \circ \eta \). We further investigate and offer some new criteria for the oscillation of equation (E). Examples are given to illustrate our results at the end of this paper.

In the sequel, all inequalities are assumed to hold eventually, that is, for all \( t \) large enough. Without loss of generality, we can deal only with the positive solutions of (E).
2. Preliminaries

In this section, we state and prove some useful lemmas, which we will use in the proof of our main results.

Lemma 1. Assume $X \geq 0, Y \geq 0, 0 \leq \beta \leq 1$. Then

$$(X + Y)^\beta \leq X^\beta + Y^\beta.$$  \hspace{1cm} (2)

Lemma 2. Assume $X \geq 0, Y \geq 0, \beta \geq 1$. Then

$$(X + Y)^\beta \leq 2^{\beta-1}(X^\beta + Y^\beta).$$  \hspace{1cm} (3)

Lemma 3 ([5], Kiguradze Lemma). Let $z(t) \in C^{n-1}([t_0, \infty))$ and $a(t)(z^{(n-1)}(t))^\gamma \in C^1([t_0, \infty))$ with $z(t) > 0$, $(a(t)(z^{(n-1)}(t))^\gamma)' \leq 0$ and not identically zero on a subinterval of $[t_0, \infty)$. Then there exist a $t_1 \geq t_0$ and an odd integer $l$, $0 < l \leq n - 1$, so that

$$(-1)^{l+j}z^{(j)}(t) > 0, \quad j = l, l + 1, \ldots, n - 1,$$
$$z^{(i)}(t) > 0, \quad i = 1, 2, \ldots, l - 1, \quad \text{when } l > 1,$$

on $[t_1, \infty)$.

J. Džurina and B. Baculíková generalization of the Philos and Staikos Lemma:

Lemma 4 ([8]). Let $z(t)$ be as in Lemma 3 and numbers $t - 1$ and $l$ be assigned to $z(t)$ by Lemma 3. Then for $0 < l < n - 1$,

$$z(t) \geq \frac{a^{1/\gamma}(t)z^{(n-1)}(t)}{(n-2)!} \int_{t_1}^{t} a^{-1/\gamma}(s)(s - t_1)^{n-2} ds,$$  \hspace{1cm} (4)

and for $l = n - 1$,

$$z(t) \geq \frac{a^{1/\gamma}(t)z^{(n-1)}(t)}{(n-2)!} \int_{t_1}^{t} a^{-1/\gamma}(s)(t - s)^{n-2} ds,$$  \hspace{1cm} (5)

where $t \geq t_1$. 


Lemma 5 ([8]). Let \( z(t) \) be as in Lemma 3 and \( a'(t) \geq 0 \). Then for any \( \lambda \in (0, 1) \) there exists some \( t_\lambda \geq t_1 \) such that

\[
z(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t)
\]

for \( t \geq t_\lambda \).

3. Main Results

In this section, we establish some oscillation criteria for (E). For the sake of convenience, we let

\[
Q(t) = \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\eta(t)))\},
\]

\[
J(t, t_1) = \int_{t_1}^{t} a^{-1/\gamma}(s) (s - t_1)^{n-2} ds,
\]

\[
J_*(t, t_1) = \int_{t_1}^{t} a^{-1/\gamma}(s) (\sigma(t) - s)^{n-2} ds.
\]

Theorem 6. Assume that \( \eta(t) \geq t, 0 < \gamma \leq 1 \) and \((\sigma^{-1}(t))' \geq \sigma_0 > 0\). Further, assume that both first-order delay differential equations

\[
w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + \rho_0^2)} (\sigma(t))^{\gamma} J_*(\sigma(t), t_1) w(\sigma(t)) = 0 \tag{6}
\]

and

\[
w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + \rho_0^2)} (\sigma(t))^{\gamma} J_*(\sigma(t), t_1) w(\sigma(t)) = 0 \tag{7}
\]

are oscillatory. Then (E) is oscillatory.

Proof. Assume to the contrary that, there exists a non-oscillatory solution \( x \) of (E). Without loss of generality, we only consider the case when \( x(t) \) is eventually positive. Then the corresponding function \( z(t) \) is positive. It follows from \((A_2), (A_4)\) and (E) that

\[
0 = \frac{a(\sigma^{-1}(t)) z^{(n-1)}(\sigma^{-1}(t))^{\gamma}'}{(\sigma^{-1}(t))'} + q(\sigma^{-1}(t)) x^{\gamma}(t)
\]
Using above in (12) implies
\[ \frac{(a(\sigma^{-1}(t))z^{(n-1)}(\sigma^{-1}(t)))'}{\sigma_0} + q(\sigma^{-1}(t))x^\gamma(t), \]  
and similarly
\[ \frac{(a(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))')}{(\sigma^{-1}(\eta(t)))'} + q(\sigma^{-1}(\eta(t)))x^\gamma(\eta(t)) \]
\[ \geq \frac{(a(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))')}{\sigma_0 \eta_0} + q(\sigma^{-1}(\eta(t)))x^\gamma(\eta(t)). \]  

Multiplying (9) with \( p_0^\gamma \) and then combining with (8), we get
\[ \frac{(a(\sigma^{-1}(t))z^{(n-1)}(\sigma^{-1}(t)))'}{\sigma_0} + \frac{p_0^\gamma(a(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))')}{\sigma_0 \eta_0} + q(\sigma^{-1}(t))x^\gamma(t) + p_0^\gamma q(\sigma^{-1}(\eta(t)))x^\gamma(\eta(t)) \leq 0. \]  

By Lemma 1 and the definition of \( z \),
\[ q(\sigma^{-1}(t))x^\gamma(t) + p_0^\gamma q(\sigma^{-1}(\eta(t)))x^\gamma(\eta(t)) \geq Q(t)[x^\gamma(t) + p_0^\gamma x^\gamma(\eta(t))] \]
\[ \geq Q(t)[x(t) + p_0 x(\eta(t))]^\gamma \]
\[ \geq Q(t)z^\gamma(t). \]  
Setting \( u(t) = a(t)[z^{(n-1)}(t)]^\gamma > 0 \) is decreasing and using (11) in (10) implies
\[ \left( \frac{u(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} u(\sigma^{-1}(\eta(t))) \right)' + Q(t)z^\gamma(t) \leq 0. \]  

Moreover, \( \left( a(t)[z^{(n-1)}(t)]^\gamma \right)' < 0 \) and there exists a \( t_1 \geq t \) and an odd integer \( l \) such that Lemma 3 holds.

If \( 0 < l < n - 1 \), then by Lemma 4
\[ z(t) \geq a^{1/\gamma}(t)z^{(n-1)}(t) \frac{J(t, t_1)}{(n-2)!}. \]  

Using above in (12) implies
\[ \left( \frac{u(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} u(\sigma^{-1}(\eta(t))) \right)' + \frac{Q(t)}{((n-2)!)^\gamma} J^\gamma(t, t_1) u(t) \leq 0. \]  

Let us denote
\[ w(t) = \frac{y(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} u(\sigma^{-1}(\eta(t))). \]
Then, from $\eta(t) \geq t$, we obtain

$$w(t) \leq u(\sigma^{-1}(t)) \left( \frac{1}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} \right).$$

That is

$$u(t) \geq \frac{\sigma_0 \eta_0}{\eta_0 + p_0} w(\sigma(t)).$$

Substituting these terms into (14), we find $w(t)$ is an eventually positive solution of

$$w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J^\gamma(\sigma(t), t_1) w(\sigma(t)) \leq 0$$

By [4, Theorem 1], the corresponding equation (6) has also positive solution, we have a contradiction.

If $l = n - 1$, then by Lemma 4

$$z(t) \geq \frac{a^{1/\gamma}(t) z^{(n-1)}(t)}{(n-2)!} \int_{t_1}^t a^{-1/\gamma}(s) (t-s)^{n-2} ds.$$

Using the last result in (12), one gets

$$\left( \frac{u(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} u(\sigma^{-1}(\eta(t))) \right)' + \frac{Q(t)}{((n-2)!)^\gamma} J^\gamma(\sigma(t), t_1) u(t) \leq 0,$$

and proceeding as above, (7) has a positive solution. This is a contradiction which completes the proof.

**Corollary 7.** Assume that $\eta(t) \geq t$, $0 < \gamma \leq 1$ and $(\sigma^{-1}(t))' \geq \sigma_0 > 0$ and

$$\liminf_{t \to \infty} \int_{\sigma(t)}^t Q(u) J^\gamma(\sigma(u), t_1) du > \frac{(\eta_0 + p_0^\gamma)((n-2)!)^\gamma}{\eta_0 \sigma_0 e}$$

(16)

and

$$\liminf_{t \to \infty} \int_{\sigma(t)}^t Q(u) J^\gamma_*(\sigma(u), t_1) du > \frac{(\eta_0 + p_0^\gamma)((n-2)!)^\gamma}{\eta_0 \sigma_0 e}$$

(17)

Then (E) is oscillatory.
Proof. According to [3, Theorem 2.1.1], the conditions (16) and (17) guarantee that equation (6) and (7) has no eventually positive solutions and therefore (E) is oscillatory.

**Theorem 8.** Let \( \eta^{-1} \) exist. Assume that \( \sigma \leq \eta(t) \leq t, \) \( 0 < \gamma \leq 1, \) \( (\sigma^{-1}(t))' \geq \sigma_0 > 0, \) and \( \eta^{-1}(t) \geq \eta_0 > 0. \) Further, assume that both first-order delay differential equations

\[ w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J_\gamma(\sigma(t), t_1) w(\eta^{-1}(\sigma(t))) = 0 \tag{18} \]

and

\[ w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J_\gamma^*(\sigma(t), t_1) w(\eta^{-1}(\sigma(t))) = 0 \tag{19} \]

are oscillatory. Then (E) is oscillatory.

**Proof.** Assume to the contrary that, there exists a non-oscillatory solution \( x \) of (E). Without loss of generality, we only consider the case when \( x(t) \) is eventually positive. Proceeding as in the proof of Theorem 6, we verify that \( w(t) \) defined by (15) satisfies

\[ w(t) \leq u(\sigma^{-1}(\eta(t))) \left( \frac{1}{\sigma_0} + \frac{p_0^\gamma}{\sigma_0 \eta_0} \right). \]

Substituting the above formulas into (14), we find out that \( w(t) \) has a positive solution of

\[ w'(t) + \frac{\eta_0 \sigma_0}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J_\gamma(\sigma(t), t_1) w(\eta^{-1}(\sigma(t))) \leq 0, \]

a contradiction. The rest of the proof is similar to that of Theorem 6 and is omitted. \( \square \)

**Corollary 9.** Let \( \eta^{-1} \) exist. Assume that \( \sigma \leq \eta(t) \leq t, \) \( 0 < \gamma \leq 1, \) \( (\sigma^{-1}(t))' \geq \sigma_0 > 0, \) and \( \eta^{-1}(t) \geq \eta_0 > 0, \)

\[ \liminf_{t \to \infty} \int_{\eta^{-1}(\sigma(u))}^{t} Q(u) J_\gamma(\sigma(u), t_1) \, du > \frac{(\eta_0 + p_0^\gamma)((n-2)!)^\gamma}{\eta_0 \sigma_0 e} \]

and
\[
\liminf_{t \to \infty} \int_{\eta^{-1}(\sigma(t))}^{t} Q(u)J^*_\gamma(\sigma(u), t_1) \, du > \frac{(\eta_0 + p_0^\gamma)((n - 2)!)^\gamma}{\eta_0 \sigma_0 e}.
\]  

(21)

Then (E) is oscillatory.

Proof. According to [3, Theorem 2.1.1], the conditions (20) and (21) guarantee that equation (18) and (19) has no eventually positive solutions and therefore (E) is oscillatory.

**Theorem 10.** Assume that \( \eta(t) \geq t, \gamma \geq 1 \) and \( (\sigma^{-1}(t))' \geq \sigma_0 > 0 \). Further, assume that both first-order delay differential equations

\[
w'(t) + \frac{\eta_0 \sigma_0 2^{1-\gamma}}{(\eta_0 + p_0^\gamma) ((n - 2)!)^\gamma} J^\gamma(\sigma(t), t_1) w(\sigma(t)) = 0
\]

and

\[
w'(t) + \frac{\eta_0 \sigma_0 2^{1-\gamma}}{(\eta_0 + p_0^\gamma) ((n - 2)!)^\gamma} J^*_\gamma(\sigma(t), t_1) w(\sigma(t)) = 0
\]

are oscillatory. Then (E) is oscillatory.

Proof. The result can be proved exactly as Theorem 6. We only replace the inequality (11) by

\[
q(\sigma^{-1}(t))x^{\gamma}(t) + p_0^\gamma q(\sigma^{-1}(\eta(t)))x^{\gamma}(\eta(t)) \geq Q(t)[x^{\gamma}(t) + p_0^\gamma x^{\gamma}(\eta(t))]
\]

\[
\geq Q(t) \frac{Q(t)}{2^{\gamma-1}} [x(t) + p_0 x(\eta(t))]^\gamma
\]

\[
\geq \frac{Q(t)}{2^{\gamma-1}} z^{\gamma}(t),
\]

which follows from Lemma 2 and the definition of \( z \).

The following result are equivalent to Theorem 8. Since they are obvious, their proofs are omitted.

**Theorem 11.** Let \( \eta^{-1} \) exist. Assume that \( \sigma \leq \eta(t) \leq t, \gamma \geq 1, (\sigma^{-1}(t))' \geq \sigma_0 > 0 \), and \( \eta^{-1}(t) \geq \eta_0 > 0 \). Further, assume that both first-order delay differential equations
\[ w'(t) + \frac{\eta_0 \sigma_0 2^{1-\gamma}}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J_\gamma^\gamma(\sigma(t), t_1) w(\eta^{-1}(\sigma(t))) = 0 \] (25)

and

\[ w'(t) + \frac{\eta_0 \sigma_0 2^{1-\gamma}}{(\eta_0 + p_0^\gamma)} \frac{Q(t)}{((n-2)!)^\gamma} J_\gamma^\gamma(\sigma(t), t_1) w(\eta^{-1}(\sigma(t))) = 0 \] (26)

are oscillatory. Then (E) is oscillatory.

Next, we present some examples to illustrate the main results.

**Example 12.** Consider the even order differential equation

\[ \left( t^{-1/3} \left[ (x(t) + p_0 x(t-3))^{(n-1)} \right]^{1/3} \right)' + \frac{\beta}{(t^{n-1})^{1/3}} x^{1/3} (t - 6) = 0, \]

\( t \geq 1, \) where \( \beta > 0. \) Then one can see that all conditions of Corollary 7 are satisfied. Hence every solution of equation (27) oscillatory.

**Example 13.** Consider the fourth order differential equation

\[ \left( t^{-1/3} \left[ (x(t) + p_0 x(t+3))^{(3)} \right]^{1/3} \right)' + \frac{\beta}{t} x^{1/3} (t/2) = 0, \quad t \geq 1, \]

where \( \beta > 0. \) Then one can see that all conditions of Corollary 9 are satisfied. Hence every solution of equation (28) oscillatory.

**References**


