THE HARMONIC ANALYSIS ASSOCIATED TO
THE CHEREDNIK-TRIMÈCHE’S
TRANSMUTATION OPERATORS ON $\mathbb{R}^d$

Khalifa Trimèche
Faculty of Sciences of Tunis
Department of Mathematics
CAMPUS, 2092 Tunis, TUNISIA

Abstract: We consider in this paper two Cherednik operators $T^k_j, T^l_j$, $j = 1, 2, 3, ..., d$, on $\mathbb{R}^d$, associated to the multiplicity functions $k, l$. First we define and study in this paper the Cherednik-Trimèche’s transmutation operator $U_{kl}$ and its dual $^tU_{kl}$. Next we study the Harmonic Analysis associated to these operators.

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1. Introduction

In [1] I. Cherednik has introduced a family of differential-difference operators that nowadays bear his name. These operators play a crucial role in the theory of Heckman-Opdam’s hypergeometric functions, which generalize the theory of Harish-Chandra’s spherical functions on Riemmann symmetric spaces (see [2,3,4]).

We consider in this paper two Cherednik operators $T^k_j$ and $T^l_j$, $j = 1, 2, ..., d$, on $\mathbb{R}^d$, associated to the multiplicity functions $k, l \in [0, \infty)$.

By using the Harmonic Analysis associated to the Cherednik operators (see [2,3,4,5,6,7]), given in Sections 2,3,4,5,6, we define and study in the other

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sections the Cherednik-Trimèche’s transmutation operator $U_{kl}$ and its dual $t_{U_{kl}}$, the Cherednik-Trimèche’s translation operator $T_{x}^{kl}$ and its dual $t_{T_{x}^{kl}}$, the Cherednik-Trimèche’s convolution product, the Cherednik-Trimèche’s heat kernel $p_{t}^{kl}(x,y)$.

2. The Cherednik operators and their eigenfunctions

We consider $\mathbb{R}^d$ with the standard basis $\{e_j; j = 1, 2, \ldots, d\}$ and the inner product $\langle ., . \rangle$ for which this basis is orthonormal.

2.1. The root system

Let $\alpha \in \mathbb{R}^d \setminus \{0\}$ and $\tilde{\alpha} = \frac{2}{\|\alpha\|^2} \alpha$. We denote by

$$r_{\alpha}(x) = x - \langle \tilde{\alpha}, x \rangle \alpha, \quad x \in \mathbb{R}^d,$$

(2.1)

the reflection on the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$. For $d = 1$, we take $\alpha = 2$.

A finite set $\mathcal{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $r_\alpha \mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$.

The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. Let $\mathbb{R}^d_{reg} = \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$.

A function $k : \mathcal{R} \to [0, +\infty[$ is called a multiplicity function, if it is invariant under the action of the reflection group $W$. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha).$$

(2.2)

2.2. The Cherednik operators

The Cherednik operators $T_{j}^{k}, j = 1, 2, \ldots, d$, on $\mathbb{R}^d$ associated with the reflection group $W$ and the multiplicity function $k$, are defined for $f$ of class $C^1$ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d_{reg}$ by

$$T_{j}^{k} f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha_j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j^{k} f(x),$$

(2.3)
where
\[ \rho_j^k = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \quad \text{and} \quad \alpha^j = \langle \alpha, e_j \rangle. \] (2.4)

The Cherednik operators form a commutative system of differential-difference operators.

For \( f \) of class \( C^1 \) on \( \mathbb{R}^d \) with compact support and \( g \) of class \( C^1 \) on \( \mathbb{R}^d \), we have for \( j = 1, 2, \ldots, d \):

\[ \int_{\mathbb{R}^d} T_j^k f(x)g(x)A_k(x)dx = -\int_{\mathbb{R}^d} f(x)(T_j^k + S_j^k)g(x)A_k(x)dx, \] (2.5)

with \( A_k \) the weight function given by
\[ \forall x \in \mathbb{R}^d, \quad A_k(x) = \prod_{\alpha \in \mathcal{R}_+} |2 \sinh(\frac{\alpha}{2}, x)|^{2k(\alpha)}, \] (2.6)

which is \( W \)-invariant and
\[ \forall x \in \mathbb{R}^d, S_j^k g(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j g(r_\alpha x). \] (2.7)

**Example 2.1.** We consider for \( d = 1 \), the root system \( \mathcal{R} = \{ \pm \alpha, \pm 2\alpha \} \), with \( \alpha = 2 \). Here \( \mathcal{R}_+ = \{ \alpha, 2\alpha \} \), and the reflection group is \( W = \mathbb{Z}_2 \). We denote by \( k \) the multiplicity function. The Cherednik operator \( T_1^k \) is defined for \( f \) of class \( C^1 \) on \( \mathbb{R} \), and \( x \) in \( \mathbb{R} \setminus \{0\} \) by
\[ T_1^k f(x) = \frac{d}{dx} f(x) + \left( \frac{2k(\alpha)}{1 - e^{-2x}} + \frac{4k(2\alpha)}{1 - e^{-4x}} \right) (f(x) - f(-x)) - \rho^k f(x) \] (2.8)
with \( \rho^k = k(\alpha) + 2k(2\alpha) \).

If we put \( k_1 = k(\alpha) + k(2\alpha), k_2 = k(2\alpha) \), the operator \( T_1^k \) takes the following form
\[ T_1^k f(x) = \frac{d}{dx} f(x) + (k_1 \coth(x) + k_2 \tanh(x))(f(x) - f(-x)) - \rho^k f(-x), \] (2.9)
with \( \rho^k = k_1 + k_2 \).

**2.3. The Opdam-Cherednik’s kernel**

We denote by \( G_\lambda^k, \lambda \in \mathbb{C}^d \), the eigenfunction of the operators \( T_j^k, j = 1, 2, \ldots, d \). It is the unique analytic function on \( \mathbb{R}^d \) which satisfies the differential-difference
system
\[
\begin{aligned}
T^k_j G^k_\lambda(x) &= i\lambda_j G^k_\lambda(x), \quad j = 1, 2, \ldots, d, \quad x \in \mathbb{R}^d, \\
G^k_\lambda(0) &= 1.
\end{aligned}
\]  
(2.10)

It is called the Opdam-Cherednik’s kernel.

**Remarks 2.1.** For \(k = 0\), we have for all \(x \in \mathbb{R}^d\), \(G^k_\lambda(x) = e^{i\langle \lambda, x \rangle}\).

The functions \(G^k_\lambda\) possess the following properties:

i) For all \(\lambda \in \mathbb{C}^d\), the function \(x \mapsto G^k_\lambda(x)\) is of class \(C^\infty\) on \(\mathbb{R}^d\).

ii) For all \(x \in \mathbb{R}^d\), the function \(\lambda \mapsto G^k_\lambda(x)\) is entire on \(\mathbb{C}^d\).

iii) For all \(x \in \mathbb{R}^d\) and \(\lambda \in \mathbb{C}^d\), we have

\[
\overline{G^k_\lambda(x)} = G^{k}_{-\bar{\lambda}}(x).
\]  
(2.11)

iv) For all \(x \in \mathbb{R}^d\) and \(\lambda \in \mathbb{C}^d\), we have

\[
|G^k_\lambda(x)| \leq G^k_{i\Im m(\lambda)}(x).
\]  
(2.12)

v) For all \(x \in \mathbb{R}^d\) and \(\lambda \in \mathbb{R}^d\), we have

\[
|G^k_\lambda(x)| \leq |W|^{1/2}.
\]  
(2.13)

vi) Let \(p\) and \(q\) be polynomials of degree \(m\) and \(n\). Then, there exists a positive constant \(M\) such that for all \(\lambda \in \mathbb{C}^d\) and \(x \in \mathbb{R}^d\), we have

\[
|p(\frac{\partial}{\partial \lambda})q(\frac{\partial}{\partial x})G^k_\lambda(x)| \leq M(1 + \|x\|)^m(1 + \|\lambda\|)^n F^k_0(x)e^{-\max_{w \in W} \Im m(w\lambda, x)}.
\]  
(2.14)

where

\[
\forall x \in \mathbb{R}^d, \quad F^k_0(x) = \frac{1}{|W|} \sum_{w \in W} G^k_0(wx).
\]  
(2.15)

**Example 2.2.** We consider for \(d = 1\), the root system \(\mathcal{R} = \{\pm \alpha, \pm 2\alpha\}\), with \(\alpha = 2\), the reflection group \(W = \mathbb{Z}_2\), the multiplicity function \(k\), the parameters \(k_1, k_2\), and the Cherednik operator \(T^k_1\), given in Example 2.1.

The Opdam-Cherednik’s kernel is given by

\[
\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, \quad G^k_\lambda(x) = \varphi^{(k_1-\frac{1}{2}, k_2-\frac{1}{2})}_\lambda(x) + \frac{1}{i\lambda - \rho^k} \frac{d}{dx} \varphi^{(k_1-\frac{1}{2}, k_2-\frac{1}{2})}_\lambda(x).
\]  
(2.16)
where \( \varphi^{(a,b)}_\lambda(x) \) is the Jacobi function of index \((a, b)\) given by
\[
\varphi^{(a,b)}_\lambda(x) = 2F_1\left(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -\sinh(x)^2\right),
\]
with \(2F_1\) the hypergeometric function of Gauss and \(\rho = a + b + 1\).

3. The intertwining operator \( V_k \) and its dual \( {}^tV_k \)

**Notation.** We denote by
- \( \mathcal{E}(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \). Its topology is defined by the semi-norms
  \[
  q_{n,K}(\varphi) = \sup_{|\mu| \leq n, x \in K} |D^\mu \varphi(x)|,
  \]
where \( K \) is a compact of \( \mathbb{R}^d \), \( n \in \mathbb{N} \), and
\[
D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_d^{\mu_d}}, \quad \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.
\]
- \( \mathcal{D}(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \), with compact support. We have
  \[
  \mathcal{D}(\mathbb{R}^d) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R}^d),
  \]
where \( \mathcal{D}_a(\mathbb{R}^d) \) is the space of \( C^\infty \)-functions on \( \mathbb{R}^d \) with support in the closed ball \( B(0, a) \) of center \( 0 \) and radius \( a \). The topology of \( \mathcal{D}_a(\mathbb{R}^d) \) is defined by the semi-norms
  \[
  p_n(\psi) = \sup_{0 \leq |\mu| \leq n, x \in B(0,a)} |D^\mu \varphi(x)|, \quad n \in \mathbb{N}.
  \]
The space \( \mathcal{D}(\mathbb{R}^d) \) is equipped with the inductive limit topology.
- \( \mathcal{S}(\mathbb{R}^d) \) the classical Schwartz space on \( \mathbb{R}^d \). Its topology is defined by the semi-norms
  \[
  Q_{\ell,n}(f) = \sup_{0 \leq |\mu| \leq n, x \in \mathbb{R}^d} (1 + \|x\|)^\ell |D^\mu f(x)|, \quad n, l \in \mathbb{N}.
  \]
- \( \mathcal{S}_2(\mathbb{R}^d) \) the generalized Schwartz space of \( C^\infty \)-functions on \( \mathbb{R}^d \) such that for \( \ell, n \in \mathbb{N} \), we have
  \[
  P_{n,\ell}(f) = \sup_{0 \leq |\mu| \leq n, x \in \mathbb{R}^d} (1 + \|x\|)^\ell (F^k_\ell(x))^{-1} |D^\mu f(x)| < +\infty,
  \]
where \( F_0(x) \) is the function given by the relation (2.15). It is topologized by means of the semi-norms \( P_{n,l}, n,l \in \mathbb{N} \).

- \( \mathcal{D}'(\mathbb{R}^d) \) the space of distributions on \( \mathbb{R}^d \). It is the topological dual of \( \mathcal{D}(\mathbb{R}^d) \).
- \( \mathcal{E}'(\mathbb{R}^d) \) the space of distributions on \( \mathbb{R}^d \) with compact support. It is the topological dual of \( \mathcal{E}(\mathbb{R}^d) \).

**Definition 3.1.**

i) The intertwining operator \( V_k \) is the unique linear topological isomorphism from \( \mathcal{E}(\mathbb{R}^d) \) onto itself satisfying the transmutations relations

\[
\forall x \in \mathbb{R}^d, T_j^k V_k(g)(x) = V_k \left( \frac{\partial}{\partial y_j} g \right)(x), j = 1, 2, \ldots, d, (3.1)
\]

and the relation

\[
V_k(g)(0) = g(0). (3.2)
\]

ii) The dual \( {}^t V_k \) of the operator \( V_k \) is defined by the following duality relation

\[
\int_{\mathbb{R}^d} {}^t V_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)A_k(x)dx, (3.3)
\]

with \( f \) in \( \mathcal{D}(\mathbb{R}^d) \) and \( g \) in \( \mathcal{E}(\mathbb{R}^d) \).

**Proposition 3.1.**

i) The operator \( {}^t V_k \) is a linear topological isomorphism from

\( -\mathcal{D}(\mathbb{R}^d) \) onto itself,

\( -\mathcal{S}_2(\mathbb{R}^d) \) onto \( \mathcal{S}(\mathbb{R}^d) \),

satisfying the transmutation relations

\[
\forall y \in \mathbb{R}^d, \ {}^t V_k((T_j^k + S_j^k)f)(y) = \frac{\partial}{\partial y} {}^t V_k(f)(y), (3.4)
\]

where \( S_j^k \) is the operator on \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( \mathcal{S}_2(\mathbb{R}^d) \) ) given by the relation (2.7).

ii) The dual \( {}^t V_k^{-1} \) of the operator \( V_k^{-1} \) satisfies the following duality relation

\[
\int_{\mathbb{R}^d} {}^t V_k^{-1}(f)(y)g(y)A_k(y)dy = \int_{\mathbb{R}^d} V_k^{-1}(g)(x)f(x)dx, (3.5)
\]

with \( f \) in \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( \mathcal{S}_2(\mathbb{R}^d) \) ) and \( g \) in \( \mathcal{E}(\mathbb{R}^d) \).

iii) For all \( f \) in \( \mathcal{D}(\mathbb{R}^d) \) we have

\[
\text{Supp} f \subset B(0,a) \Rightarrow \text{Supp} {}^t V_k(f) \subset B(0,a), (3.6)
\]

where \( B(0,a) \) is the closed ball of center 0 and radius \( a > 0 \).
Remarks 3.1. From the relations (2.3),(3.1),(3.4) we deduce that the operators $V_0$ and $tV_0$ are the identity operators.

4. The hypergeometric Fourier transform associated with the Cherednik operators

**Notation.** For $a > 0$, we denote by $PW(\mathbb{C}^d)_a$ (resp. $\mathcal{P}W(\mathbb{C}^d)_a$) the spaces of functions $h$ which are entire on $\mathbb{C}^d$ and satisfying
\[
\forall m \in \mathbb{N}, s_m(h) = \sup_{\lambda \in \mathbb{C}^d}(1 + \|\lambda\|^m e^{a\|\text{Im}\lambda\|}|h(\lambda)| < \infty.
\]
(resp. $\exists m \in \mathbb{N}, \sigma_m(h) = \sup_{\lambda \in \mathbb{C}^d}(1 + \|\lambda\|^{-m} e^{a\|\text{Im}\lambda\|}|h(\lambda)| < \infty.)$

Their topologies is given by the semi-norms $s_m, m \in \mathbb{N}$, (resp. $\sigma_m, m \in \mathbb{N}$).

- We consider the spaces $PW(\mathbb{C}^d)$ (resp. $\mathcal{P}W(\mathbb{C}^d)$) of the entire functions on $\mathbb{C}^d$ which are rapidly decreasing (resp. slowly increasing) and of exponential type. We have
\[
PW(\mathbb{C}^d) = \bigcup_{a > 0} PW(\mathbb{C}^d)_a \quad \text{(resp. $\mathcal{P}W(\mathbb{C}^d) = \bigcup_{a > 0} \mathcal{P}W(\mathbb{C}^d)_a$)}.
\]
They are equipped with the inductive limit topology.

**Definition 4.1.** The hypergeometric Fourier transform $\mathcal{H}^k$ is defined for all function $f$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by
\[
\forall \lambda \in \mathbb{C}^d, \quad \mathcal{H}^k(f)(\lambda) = \int_{\mathbb{R}^d} f(x) G^k_\lambda(x) A_k(x) dx. \quad (4.1)
\]

**Theorem 4.1.** The hypergeometric Fourier transform $\mathcal{H}^k$ is a topological isomorphism from
- $\mathcal{D}(\mathbb{R}^d)$ onto $PW(\mathbb{C}^d)$,
- $\mathcal{S}_2(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

The inverse transform $(\mathcal{H}^k)^{-1}$ is given by
\[
\forall x \in \mathbb{R}^d, (\mathcal{H}^k)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda) G^k_{\lambda}(-x) C_k(\lambda) d\lambda, \quad (4.2)
\]
where for all $\lambda \in \mathbb{C}^d$,
- For $k \in (0, \infty)$
\[
C_k(\lambda) = c \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(-i\langle \lambda, \alpha \rangle + \frac{1}{2} k(\alpha) + k(\alpha)) \Gamma(i\langle \lambda, \alpha \rangle + k(\alpha) + k(\alpha) + 1)}{\Gamma(-i\langle \lambda, \alpha \rangle + \frac{1}{2} \Gamma(\alpha) + \Gamma(\frac{\alpha}{2}))(i\langle \lambda, \alpha \rangle + \frac{1}{2} \Gamma(\frac{\alpha}{2}) + 1)}, \quad (4.3)
\]
with $c$ a normalising constant.

- For $k = 0$,
\[ C_k(\lambda) = 1. \]  (4.4)

**Definition 4.2.** The hypergeometric Fourier transform $H^k$ is defined for $S$ in $\mathcal{E}'(\mathbb{R}^d)$ by
\[ \forall \, \lambda \in \mathbb{R}^d, \quad H^k(S)(\lambda) = \langle S, G^k_\lambda \rangle. \]  (4.5)

**Theorem 4.2.** The transform $H^k$ is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{PW}(\mathbb{C}^d)$.

5. The hypergeometric translation operator and its dual and the hypergeometric convolution product associated with the Cherednik operators

5.1. The hypergeometric translation operator and its dual

**Definition 5.1.** The hypergeometric translation operator $T^k_x$, $x \in \mathbb{R}^d$, is defined on $\mathcal{E}(\mathbb{R}^d)$ by
\[ \forall \, y \in \mathbb{R}^d, T^k_x(f)(y) = (V_k)_x((V_k)_y[V^{-1}_k(f)(x + y)]. \]  (5.1)

**Proposition 5.1.** The operator $T^k_x$, $x \in \mathbb{R}^d$, satisfies the following properties:

i) For all $x \in \mathbb{R}^d$, the operator $T^k_x$, is continuous from $\mathcal{E}(\mathbb{R}^d)$ into itself.
ii) For all $f$ in $\mathcal{E}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have
\[ T^k_x(f)(0) = f(x), \text{ and } T^k_x(f)(y) = T^k_y(f)(x). \]  (5.2)

iii) For all $x, y \in \mathbb{R}^d$, and $\lambda \in \mathbb{C}^d$, we have the product formula
\[ T^k_x(G^k_\lambda)(y) = G^k_\lambda(x)G^k_\lambda(y), \]  (5.3)
where $G^k_\lambda$, the opdam-cherednick kernel given by (2.10).

**Definition 5.2.** For each $x \in \mathbb{R}^d$, the dual of the hypergeometric translation operator $T^k_x$, is the operator $^tT^k_x$ defined on $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by
\[ \forall \, y \in \mathbb{R}^d, ^tT^k_x(f)(y) = (V_k)_x(^tV^{-1}_k)_y[^tV_k(f)(y - x)]. \]  (5.4)
**Proposition 5.2.** We give in the following the properties of the operator $t^tT^k_x$.

i) For all $x \in \mathbb{R}^d$, the operator $t^tT^k_x$ is continuous from

- $\mathcal{D}(\mathbb{R}^d)$ into itself,
- $S_2(\mathbb{R}^d)$ into itself.

ii) The operator $t^tT^k_x$, $x \in \mathbb{R}^d$, is related to the operator $T^k_x$, $x \in \mathbb{R}^d$, by the following relation

\[
\int_{\mathbb{R}^d} T^k_x(g)(y)f(y)\Lambda_k(y)dy = \int_{\mathbb{R}^d} g(z)t^tT^k_x(f)(z)\Lambda_k(z)dz,
\]

with $g$ in $\mathcal{E}(\mathbb{R}^d)$, and $f$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$).

iii) For all $f$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) and $x \in \mathbb{R}^d$, we have

\[
\forall \lambda \in \mathbb{R}^d, \mathcal{H}^k(t^tT^k_x(f))(\lambda) = G^k_\lambda(x)\mathcal{H}^k(f)(\lambda).
\]

Thus from the relation (4.2) we have

\[
\forall y \in \mathbb{R}^d, t^tT^k_x(f)(y) = \int_{\mathbb{R}^d} G^k_\lambda(x)G^k_\lambda(-y)\mathcal{H}^k(f)(\lambda)\Lambda_k(\lambda)d\lambda.
\]

iv) For all $f$ in $\mathcal{D}(\mathbb{R}^d)$ with support in the closed ball $B(0,a)$ of center $o$ and radius $a > 0$, and $x \in \mathbb{R}^d$, we have

\[
\text{Supp} t^tT^k_x(f) \subset B(0,a + \|x\|).
\]

5.2. The hypergeometric convolution product

**Definition 5.3.** The hypergeometric convolution product $f \ast_k g$ of the functions $f,g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) is defined by

\[
\forall x \in \mathbb{R}^d, f \ast_k g(x) = \int_{\mathbb{R}^d} t^tT^k_x(f)(y)g(y)\Lambda_k(y)dy.
\]

**Proposition 5.3.** The convolution product $\ast_k$ satisfies the following properties:

i) For all $f,g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) the function $f \ast_k g$ belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$).

ii) For all $f,g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$), we have

\[
\forall \lambda \in \mathbb{R}^d, \mathcal{H}^k(f \ast_k g)(\lambda) = \mathcal{H}^k(f)(\lambda)\mathcal{H}^k(g)(\lambda).
\]
iii) This convolution product is commutative and associative.
iv) For all \( f, g \) in \( D(\mathbb{R}^d) \) (resp. \( S_2(\mathbb{R}^d) \)), we have
\[
t V_k(f *_k g) = t V_k(f) * t V_k(g),
\]
where \(*_k\) is the classical convolution product on \( \mathbb{R}^d \).

6. The heat kernel associated to the Cherednik operators

Definition 6.1. Let \( t > 0 \). The heat kernel \( p^k_t(x, y) \) associated with the Cherednik operators, is defined for all \( x, y \in \mathbb{R}^d \), by
\[
p^k_t(x, y) = \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2 + \|\rho^k\|^2)} G^k_\lambda(x) G^k_\lambda(-y) C_k(\lambda) d\lambda.
\]

Notation. We denote by:
- \( H_k \) the heat operator associated with the Cherednik operator given by
\[
H_k = \mathcal{L}_k - \frac{\partial}{\partial t} - \|\rho^k\|^2,
\]
where \( \mathcal{L}_k \) is the Heckman-Opdam Laplacian defined for \( f \) of class \( C^2 \) on \( \mathbb{R}^d \) by
\[
\mathcal{L}_k f = \sum_{j=1}^{d} (T^k_j)^2(f).
\]
- \( E^k_t, t > 0 \), the fundamental solution of the operator \( H_k \) given by
\[
\forall x \in \mathbb{R}^d, \quad E^k_t(x) = p^k_t(x, 0).
\]

Proposition 6.1. i) For all \( t > 0 \), the function \( E^k_t \) belongs to \( S_2(\mathbb{R}^d) \).
ii) For all \( t > 0 \), we have
\[
\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(E^k_t)(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho^k\|^2)}.
\]
iii) The function \( (x, t) \rightarrow E^k_t(x) \) is strictly positive on \( \mathbb{R}^d \times (0, \infty) \).
iv) For all \( t > 0 \), we have
\[
\int_{\mathbb{R}^d} E^k_t(x) A_k(x) dx = 1.
\]
v) We have
\[
H_k E^k_t(x) = 0, \quad \text{on } \mathbb{R}^d \times (0, \infty).
\]
Proposition 6.2. i) For all $t > 0$ and $x \in \mathbb{R}^d$, the function $y \to p^k_t(x, y)$ belongs to $S_2(\mathbb{R}^d)$.

ii) For all $t > 0$ and $x, y \in \mathbb{R}^d$, we have

$$p^k_t(x, y) = t^T_k x (E^k_t(y)). \quad (6.8)$$

iii) The function $p^k_t(x, y)$ is strictly positive on $\mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$.

iv) For all $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} p^k_t(x, y) A_k(y) dy = 1. \quad (6.9)$$

v) For all $y \in \mathbb{R}^d$, the function $(x, t) \to p^k_t(x, y)$ satisfies

$$H_k p^k_t(x, y) = 0, \text{ on } \mathbb{R}^d \times (0, \infty). \quad (6.10)$$

Remark 6.1. To give the new results of this paper, we consider a second multiplicity function $l : \mathcal{R} \to [0, \infty]$. We consider also the cherednik operators $T^l_j$, $j = 1, 2, \ldots, d$, the Opdam-cherednik’s kernel $G^l_{\lambda}$, the transmutation operators $V_l$ and its dual $t^l V_l$, the hypergeometric Fourier transform $\mathcal{H}^l$, the hypergeometric translation operator $T^l_x$ and its dual $t^l T^l_x$, the fundamental solution $E^l_t$ of the operator $H_l$ and of the heat kernel $p^l_t(x, y)$.

7. The Cherednik-Trimèche’s transmutation operator $U_{kl}$ and its dual $t^l U_{kl}$

Definition 7.1. The Cherednik-Trimèche’s transmutation operator $U_{kl}$ is defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall x \in \mathbb{R}^d, U_{kl}(f)(x) = V_k \circ V_l^{-1}(f)(x). \quad (7.1)$$

By using the properties of the transmutation operators $V_k$ and $V_l$ given in Section 3, we obtain the following properties of the operator $U_{kl}$.

Theorem 7.1. i) For $k = l$, we have

$$U_{kl} = Id. \quad (7.2)$$
ii) The operator $U_{kl}$ is the unique topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the condition

$$U_{kl}(f)(0) = f(0). \quad (7.3)$$

iii) The inverse operator $U_{kl}^{-1}$ is given for $f$ in $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall x \in \mathbb{R}^d, \quad U_{kl}^{-1}(f)(x) = V_l \circ V_k^{-1}(f)(x) = U_{lk}(f)(x). \quad (7.4)$$

iv) The operator $U_{kl}$ satisfies for all $f$ in $\mathcal{E}(\mathbb{R}^d)$ the following transmutation relations

$$\forall x \in \mathbb{R}^d, \quad T^k_j(U_{kl}(f))(x) = U_{kl}(T^l_j(f))(x), \quad j = 1, 2, \ldots, d. \quad (7.5)$$

v) We have

$$\forall \lambda \in \mathbb{C}^d, \forall x \in \mathbb{R}^d, \quad U_{kl}(G^\lambda_k(x)) = G^\lambda(x). \quad (7.6)$$

vi) We have

$$U_{kl}(1) = 1. \quad (7.7)$$

**Definition 7.2.** The dual of the Cherednik-Trimèche’s transmutation operator $U_{kl}$ is the operator $^tU_{kl}$ defined on $D(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) by

$$\forall y \in \mathbb{R}^d, \quad ^tU_{kl}(g)(y) = ^tV_l^{-1} \circ ^tV_k(g)(y). \quad (7.8)$$

The properties of the operator $^tV_k$ and $^tV_l$, given in Section 3, imply the following properties of the operator $^tU_{kl}$.

**Theorem 7.2.** i) For $k = l$, we have

$$^tU_{kl} = Id. \quad (7.9)$$

ii) The operator $^tU_{kl}$ is a topological isomorphism from $D(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) onto itself. iii) The inverse operator $^tU_{kl}^{-1}$ is given for $g$ in $D(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) by

$$\forall y \in \mathbb{R}^d, \quad ^tU_{kl}^{-1}(g)(y) = ^tV_k^{-1} \circ ^tV_l(g)(y) = ^tU_{lk}(g)(y). \quad (7.10)$$

iv) The operator $^tU_{kl}$ satisfies for all $g$ in $D(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) the following transmutation relation

$$\forall y \in \mathbb{R}^d, \quad ^tU_{kl}((T^k_j + S^k_j)(f))(y) = (T^l_j + S^l_j)(^tU_{kl}(g))(y). \quad (7.11)$$
**Proposition 7.1.** The operators $U_{kl}$ and $t^k U_{kl}$ are related for $f$ in $\mathcal{E}(\mathbb{R}^d)$ and $g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) by the following duality relation

$$
\int_{\mathbb{R}^d} U_{kl}(f)(x)g(x)A_k(x)dx = \int_{\mathbb{R}^d} t^k U_{kl}(g)(y)f(y)A_l(y)dy. \tag{7.12}
$$

**Corollary 7.1.** The operators $t^k U_{kl}$ satisfies for all $g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) the following expression:

$$
\forall y \in \mathbb{R}^d, \quad t^k U_{kl}(g)(y) = (\mathcal{H}^l)^{-1} \circ \mathcal{H}^k(g)(y). \tag{7.13}
$$

**Proof.** From the relations (7.12),(7.6), we have

$$
\forall \lambda \in \mathbb{R}^d, \quad H^k(g)(\lambda) = H^l(t^k U_{kl}(g))(\lambda).
$$

We deduce (7.13) from this relation and Theorem 4.2. \[\square\]

8. The Cherednik-Trimèche’s translation operator and its dual

**Definition 8.1.** For $x, y \in \mathbb{R}^d$, the Cherednik-Trimèche’s translation operator $T_{x}^{kl}$ is defined for all $f$ in $\mathcal{E}(\mathbb{R}^d)$ by

$$
T_{x}^{kl}(f)(y) = (V_l)_x((V_k)_y[V_k^{-1}](f)(x + y)]. \tag{8.1}
$$

**Definition 8.2.** For all $x, y \in \mathbb{R}^d$, the dual of the Cherednik-Trimèche’s translation operator $t^k T_x^{kl}$ is defined for all $g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) by

$$
t^k T_x^{kl}(g)(z) = (V_l)_x(t^k V^{-1}_k)_z[t^k V_k(g)(z - x)]. \tag{8.2}
$$

From the properties of the operator $V_l, V_k$ and $t^k V_k$ given in Section 3, we obtain the following properties of the operators $T_{x}^{kl}$ and $t^k T_x^{kl}$.

**Proposition 8.1.** i) For all $f$ in $\mathcal{E}(\mathbb{R}^d)$, the function $T_{x}^{kl}(f)(y)$ is of class $C^\infty$ on $\mathbb{R}^d$ with respect to the variables $x$ and $y$. ii) For all $g$ in $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) the function $t^k T_{x}^{kl}(g)(z)$ is of class $\mathbb{R}^d$ with respect to the variables $x$, and belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $S_2(\mathbb{R}^d)$) with respect to the variable $z$. More precisely if the support of $g$ is contained in the ball of center 0 and radius $a > 0$, and $x \in \mathbb{R}^d$, we have

$$
\text{Supp}^t T_{x}^{kl}(g)(z) \subset B(0, a + \|x\|), \tag{8.3}
$$
iii) For all \( f \) in \( \mathcal{E}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), we have
\[
\mathcal{T}_x^{kl}(f)(0) = V_l \circ V_k^{-1}(f)(x).
\] (8.4)

iv) For all \( g \) in \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( S_2(\mathbb{R}^d) \)) and \( z \in \mathbb{R}^d \), we have
\[
\mathcal{t}^{T}_0^{kl}(g)(z) = g(z).
\] (8.5)

v) For all \( x, y \in \mathbb{R}^d \) and \( \lambda \in \mathbb{C}^d \), we have
\[
\mathcal{T}_x^{kl}(G_\lambda^k(y)) = G_\lambda^l(x)G_\lambda^k(y).
\] (8.6)

**Proposition 8.2.** The operators \( \mathcal{T}_x^{kl} \) and \( \mathcal{t}_x^{T^{kl}} \) are related for all \( f \) in \( \mathcal{E}(\mathbb{R}^d) \) and \( g \) in \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( S_2(\mathbb{R}^d) \)) by the following duality relation:
\[
\int_{\mathbb{R}^d} \mathcal{T}_x^{kl}(f)(y)g(y)A_k(y)dy = \int_{\mathbb{R}^d} f(z)\mathcal{t}_x^{T^{kl}}(g)(z)A_k(z)dz.
\] (8.7)

**Corollary 8.1.** For all \( x \in \mathbb{R}^d \), the dual \( \mathcal{t}_x^{T^{kl}} \) of the Cherednik-Trimèche’s translation operator, satisfies for all \( g \) in \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( S_2(\mathbb{R}^d) \)) the following relations:

i) \( \forall \, \lambda \in \mathbb{R}^d, \quad \mathcal{H}_k(\mathcal{t}_x^{T^{kl}}(g)(\lambda)) = G_\lambda^l(x)\mathcal{H}_k(g)(\lambda), \) (8.8)

ii) \( \forall \, z \in \mathbb{R}^d, \quad \mathcal{t}_x^{T^{kl}}(g)(z) = \int_{\mathbb{R}^d} G_\lambda^l(x)G_\lambda^k(-z)\mathcal{H}_k(g)(\lambda)C_k(\lambda)d\lambda. \) (8.9)

**Proof.** i) By applying thr relation (8.7) with \( f(z) = G_\lambda^k(z), \, z \in \mathbb{R}^d, \, \lambda \in \mathbb{R}^d \), we obtain
\[
\int_{\mathbb{R}^d} G_\lambda^k(z)\mathcal{t}_x^{T^{kl}}(g)(z)A_k(z)dz = \int_{\mathbb{R}^d} \mathcal{T}_x^{kl}(G_\lambda^k(y))g(y)A_k(y)dy.
\]

We deduce relation (8.8) from this relation and relations (8.6),(4.1).

ii) We deduce relation (8.9) from relations (8.8),(4.2),(2.14) and Theorem 4.2. 

\[\Box\]
9. The Cherednik-Trimèche’s convolution product

**Definition 9.1.** The Cherednik-Trimèche’s convolution product of the functions \(f, g\) in \(D(\mathbb{R}^d)\) (resp. \(S_2(\mathbb{R}^d)\)) is the function \(f \ast_{kl} g\) defined by
\[
\forall \ y \in \mathbb{R}^d, \quad f \ast_{kl} g(y) = \int_{\mathbb{R}^d} \mathcal{T}_x^{kl}(f)(y) g(x) A_k(x) \, dx. \tag{9.1}
\]

**Proposition 9.1.** i) For all functions \(f, g\) in \(D(\mathbb{R}^d)\) (resp. \(S_2(\mathbb{R}^d)\)) the function \(f \ast_{kl} g\) belongs to \(D(\mathbb{R}^d)\) (resp. \(S_2(\mathbb{R}^d)\)).

ii) For all \(f, g\) in \(D(\mathbb{R}^d)\) (resp. \(S_2(\mathbb{R}^d)\)), we have the following relations:
\[
\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(f \ast_{kl} g)(\lambda) = \mathcal{H}^l(g \ast_{kl} f)(\lambda), \tag{9.2}
\]
\[
\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}_k^k(f \ast_{kl} g)(\lambda) = \mathcal{H}_k^k(f)(\lambda) \mathcal{H}_l^l(g)(\lambda). \tag{9.3}
\]

**Proposition 9.2.** For all functions \(f, g\) in \(D(\mathbb{R}^d)\) (resp. \(S_2(\mathbb{R}^d)\)), we have
\[
\forall \ y \in \mathbb{R}^d, \quad \mathcal{V}_x^k(f \ast_{kl} g)(y) = \mathcal{V}_x^k(f)^* \mathcal{V}_x^l(g)(y), \tag{9.4}
\]
where \(*\) is the classical convolution product of functions on \(\mathbb{R}^d\).

**Proof.** We deduce relation (9.4) from relations (9.1) and (8.2). \(\square\)

10. The Cherednik-Trimèche’s heat kernel

**Definition 10.1.** For all \(x, y \in \mathbb{R}^d\) and \(t \in (0, \infty)\), we define the Cherednik-Trimèche’s heat kernel \(p_t^{kl}(x, y)\) by
\[
p_t^{kl}(x, y) = \mathcal{T}_x^{kl}(E_t^k)(y), \tag{10.1}
\]
where \(E_t^k(x)\) is the fundamental solution of the operator \(H_k\) given by the relation (6.4).

**Proposition 10.1.** i) For all \(x \in \mathbb{R}^d, \ t \in (0, \infty)\) and \(\lambda \in \mathbb{R}^d\), we have
\[
\mathcal{H}^k(p_t^{kl}(x, .))(\lambda) = e^{-t(||\lambda||^2+||p^{kl}_t||^2)} G^d_{\lambda}(x). \tag{10.2}
\]
ii) For all \(x, y \in \mathbb{R}^d\) and \(t \in (0, \infty)\), we have
\[
p_t^{kl}(x, y) = \int_{\mathbb{R}^d} e^{-t(||\lambda||^2+||p^{kl}_t||^2)} G^d_{\lambda}(x) G^k_{\lambda}(y) C_k(\lambda) \, d\lambda. \tag{10.3}
\]
Proof. i) From the relations (10.1), (8.7) we have
\[ \forall (x, t) \in \mathbb{R}^d \times (0, \infty), \lambda \in \mathbb{R}^d, \mathcal{H}^k(p_t^{kl}(x, .))(\lambda) = G^l(\lambda) \mathcal{H}^k(E^k_t)(\lambda). \]
We deduce relation (10.2) from relation (6.5).

ii) The relations (10.1), (10.2), (8.9) (8.10) imply relation (10.3).

\[ \square \]

Proposition 10.2. For \( x \in \mathbb{R}^d \) and \( t \in (0, \infty) \), we have
\[ \forall y \in \mathbb{R}^d, tU_{kl}(p_t^{kl}(x, .))(y) = e^{-t(\|\rho^k\|^2 - \|\rho^l\|^2)} p_t(x, y). \quad (10.4) \]

Proof. From the relations (7.13), (10.2), we have for all \( y \in \mathbb{R}^d \),
\[ tU_{kl}(p_t^{kl}(x, .))(y) = (\mathcal{H}^l)^{-1} [e^{-t(\|\lambda\|^2 + \|\rho^k\|^2)} G^l(\lambda)(x)](y) \]
\[ = e^{-t(\|\rho^k\|^2 - \|\rho^l\|^2)} (\mathcal{H}^l)^{-1} [e^{-t(\|\lambda\|^2 + \|\rho^l\|^2)} G^l(\lambda)(x)](y). \]
By using the relation (4.2) we obtain for all \( y \in \mathbb{R}^d \),
\[ tU_{kl}(p_t^{kl}(x, .))(y) = e^{-t(\|\rho^k\|^2 - \|\rho^l\|^2)} \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2 + \|\rho^l\|^2)} G^l(\lambda)(x) G^l(-y) C^l(\lambda) d\lambda. \]
Thus the relation (6.1) implies
\[ \forall y \in \mathbb{R}^d, tU_{kl}(p_t^{kl}(x, .))(y) = e^{-t(\|\rho^k\|^2 - \|\rho^l\|^2)} p_t(x, y). \]

\[ \square \]

In coming papers we plan to study:

1. The Harmonic Analysis associated to the Cherednik-Trimèche’s transmutation operators on \( \mathbb{R}^d \) in the \( W \)-invariant case.

2. Applications of the Harmonic Analysis associated to the Cherednik-Trimèche’s operators on \( \mathbb{R}^d \).

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References


