ON THE FAITHFULNESS OF JONES-WENZL REPRESENTATION OF THE BRAID GROUP $B_4$

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Abstract: We consider the Jones-Wenzl representation $\rho_{4}^{(k,r)}$ of the braid group $B_4$. For each pair of integers $(k,r)$ with $r \geq k + 1$, we exclude a family of words in the generators from belonging to the kernel of $\rho_{4}^{(k,r)}$. Then, we prove that $\rho_{4}^{(k,r)}$ is not faithful for any even integer $r$.

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1. Introduction

The linear representation of the braid group on $n$ strings, namely the Burau representation, has been shown that it is not faithful for $n \geq 6$ \cite{6}. Since then, irreducible representations of the braid group have been constructed to attack the question of the faithfulness of the braid group $B_n$ or its normal subgroup $P_n$. An attempt was done on the Burau representation of the braid group on four strings \cite{1}.

In \cite{4}, H. Haidar and M. Abdulrahim constructed a representation of $P_3$, the pure braid group on three strands. They studied its irreducibility and unitarity as a tool to study its faithfulness in the future.

We consider Jones-Wenzl representation of the braid group $B_n$. This new family of representations $\rho$ of the braid groups were discovered by V. Jones in 1983. While many sought to bend Jones’ theory toward classical topological
objectives, M. Freedman, M. Larsen, and Z. Wang have found that the relation between the Jones polynomial and physics allows potentially realistic models of quantum computation to be created. For more details (see [3]).

A feature of Jones representation is the two-eigen value property. That is, the image of each braid generator has only two distinct eigenvalues $\{-1, q\}$. This property was used in 2002, in [2], in the identification of the closed images of the irreducible components of Jones representation. Also, the closed images of the special unitarity groups $SU(N)$ were identified completely for the general case.

**Theorem 1.** Fix an integer $r \geq 5$, $r \neq 6, 10$, $n \geq 3$ or $r=10$, $n \geq 5$. Let

$$\rho_{n}^{(2,r)} = \bigoplus_{\lambda \in \Lambda_n^{(2,r)}} B_n \to \prod_{\lambda \in \Lambda_n^{(2,r)}} U(\lambda)$$

be the unitary Jones representation of the $n$-strand braid group $B_n$. Then the closed image $\rho_{n}^{(2,r)}(B_n)$ contains $\prod_{\lambda \in \Lambda_n^{(2,r)}} SU(\lambda)$.

The original motivation for studying Jones representation is for quantum computation. The special case $r = 5$ has already been used to show that the $SU(2)$ Witten-Chern-Simons modular functor at the fifth root of unity is universal for quantum computation. Combining that paper with the above result, it was concluded that the $SU(2)$ Witten-Chern-Simons modular functor at an $r^{th}$ root of unity is universal for quantum computation if $r \neq 3, 4, 6$. For more details (see [2]).

The unitary Jones-Wenzl representation $\rho_{n}^{(k,r)}$ of the braid group $B_n$ was defined for a pair of integers $(k, r)$ with $r \geq k + 1$.

In our work, we consider Jones representation $\rho_{n}^{(k,r)}$ for $n = 4$. We determine the explicit form of the matrices of the generators of $B_4$ for $k = 3$ and $r = 8$. We then approach the question of the faithfulness of the unitary Jones representation of $B_4$. Our first result is to exclude a family of words in the generators from belonging to the kernel of the considered representation $\rho_{4}^{(k,r)}$, (Theorem 2). Also, we prove that $\rho_{4}^{(k,r)}$ is not faithful for any even integer $r$, (Theorem 3).

### 2. Preliminaries

**Definition 1.** ([5]) The braid group on $n$ strings, $B_n$, is the abstract group with presentation:
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\[ B_n = \left\{ \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \ldots, n-2 \right\}. \]

**Definition 2.** Let \( GL(V) \) be the general linear group of the vector space \( V \) over the field \( K \). A representation \( \rho \) of a group \( G \) on \( V \) is said to be faithful if the group homomorphism \( \rho : G \to GL(V) \) is injective.

**Definition 3.** ([2]) The Hecke algebra \( H_n(q) \) of type \( A \) is the (finite dimensional) complex algebra generated by \( e_1, \ldots, e_{n-1} \) such that

1. \( e_i^2 = e_i \),
2. \( e_i e_{i+1} e_i - 2 e_i e_{i+1} - 2 e_i = e_{i+1} e_i e_{i+1} - 2 e_{i+1}, \)
3. \( e_i e_j = e_j e_i \text{ if } |i - j| \geq 2. \)

Here \([2] = q^{1/2} + q^{-1/2} = 2 \cos \frac{\pi}{r} (r \in \mathbb{Z}).\)

**Definition 4.** Let \( \pi \) be a representation of the Hecke algebra \( H_n(q) \) generated by \( e_i, i = 1, 2, \ldots, n - 1 \), on a Hilbert space. The representation \( \pi \) is called a \( \mathbb{C}^* \) representation if each \( \pi(e_i) \) is self-adjoint.

### 3. Jones-Wenzl Representation

Hecke algebra representations of the braid groups in the root of unity case are indexed by two parameters: a compact Lie group and an integer \( l \geq 1 \), called the level of the theory. The cases of Jones and Wenzl representations correspond to the special unitary groups \( SU(k), k \geq 2 \). For each pair of integers \( (k, r) \) with \( r \geq k + 1 \), there is a unitary representation of the braid groups with level \( l = r - k \).

To describe the Jones-Wenzl representation, we let \( q = e^{\pm \frac{2\pi i}{r}} \) and \( [m] \) be the quantum integer \( \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} \).

**Lemma 1.** Each \( \mathbb{C}^* \) representation of the Hecke algebra \( H_n(q) \) gives rise to a unitary representation of the braid group \( B_n \) by the formula:

\[ \rho(\sigma_i) = q - (1 + q)\pi(e_i). \]
Jones-Wenzl $\mathbb{C}^*$ representation of $H_n(q)$ are reducible. Their irreducible constituents are indexed by Young diagrams. A Young diagram with $n$ boxes is a diagram of a partition of $n$ such that $\lambda = [\lambda_1, ..., \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$. $\sum_{i=1}^{k} \lambda_i = n$. Let $\lambda$ be a Young diagram with $n$ boxes. A Young diagram with $n$ boxes is a diagram of a partition of $n$ such that $\lambda = [\lambda_1, ..., \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$. $\sum_{i=1}^{k} \lambda_i = n$.

Definition 5. ([2]) Suppose $t$ is a standard tableau with $n$ boxes, and $m_1$ and $m_2$ are two entries in $t$. Suppose $m_i$ appears in row $r_i$ and column $c_i$ of $t$.

1. Set $d_{t,m_1,m_2} = (c_1 - c_2) - (r_1 - r_2)$.
2. Set $\alpha_{t,i} = \frac{[d_{t,i,i+1}]}{[d_{t,i,i+1}]}$ if $[d_{t,i,i+1}] \neq 0$ and $\beta_{t,i} = \sqrt{\alpha_{t,i}(1 - \alpha_{t,i})}$.
3. A Young diagram $\lambda = [\lambda_1, ..., \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$ is $(k,r)$-admissable if $\lambda_1 - \lambda_k \leq r - k$.
4. Suppose $t$ is a standard tableau of shape $\lambda$ with $n$ boxes, let $t^{(i)}$ ($1 \leq i \leq n$) be the standard tableaux obtained from $t$ by deleting boxes with entries $n, n-1, ..., n-i+1$. A standard tableau $t$ is $(k,r)$-admissable if the shape of each tableau $t^{(i)}$ is a $(k,r)$-admissable Young diagram.

The irreducible sectors of the Jones-Wenzl representations of the Hecke algebras $H_n(q)$ are indexed by the pair $(k,r)$ and a $(k,r)$-admissable Young diagram $\lambda$ with $n$ boxes. Note that $\lambda$ is allowed to have empty rows.

To construct a $\mathbb{C}^*$ representation $\pi^{(k,r)}_\lambda$ of the Hecke algebra $H_n(q)$, the $(k,r)$-admissable standard tableaux $t$ of shape $\lambda$ are considered. By interchanging the entries $i$ and $i + 1$ in $t$, the tableau $s_i(t)$ is obtained.

If $s_i(t)$ is also $(k,r)$-admissable standard tableau, then

$$\pi^{(k,r)}_\lambda(e_i)(v_t) = \alpha_{t,i}v_t + \beta_{t,i}v_{s_i(t)},$$

where $\{v_t\}$ is the basis of the complex vector space $V^{(k,r)}_\lambda$.

If $s_i(t)$ is not $(k,r)$-admissable standard tableau, set $\beta_{t,i} = 0$. In this case, $\alpha_{t,i}$ is either 0 or 1.

Consequently, $\pi^{(k,r)}_\lambda(e_i)$ is a matrix consisting of only $2 \times 2$ blocks

$$\begin{pmatrix} \alpha_{t,i} & \beta_{t,i} \\ \beta_{t,i} & 1 - \alpha_{t,i} \end{pmatrix}$$

and $1 \times 1$ blocks 0 or 1.
The restriction of $\pi^{(k,r)}_{\lambda}$ to $B_n$ is denoted by $\rho^{(k,r)}_{\lambda}$. Direct calculations show that $\rho^{(k,r)}_{\lambda}(\sigma_i)$ is a matrix consisting of only $2 \times 2$ blocks

$$\rho_{\lambda}(\sigma_i) = \begin{pmatrix} q - (1 + q)\alpha_{t,i} & -\beta_{t,i} \\ -\beta_{t,i} & q - (1 + q)(1 - \alpha_{t,i}) \end{pmatrix}$$

and $1 \times 1$ blocks $q$ or $-1$.

**Definition 6.** ([2]) Given a pair of integers $(k, r)$ with $r \geq k + 1$. Let $\Lambda^{(k,r)}_n$ be the set of all $(k, r)$-admissible Young diagrams with $n$ boxes. The Jones-Wenzl representation of the braid group $B_n$ is:

$$\rho^{(k,r)}_n = \bigoplus_{\lambda \in \Lambda^{(k,r)}_n} \rho^{(k,r)}_{\lambda} : B_n \to \prod_{\lambda \in \Lambda^{(k,r)}_n} U(\lambda).$$

Here we write $U(\lambda)$ for the unitary group of the Hilbert space $V^{(k,r)}_{\lambda}$ with the orthonormal basis $\{v_i\}$.

### 4. Jones-Wenzl Representation for $k=3$ and $r=8$

Using the explicit description of Jones-Wenzl representation in Section 3, we write explicitly the $9 \times 9$ matrices of the generators of $B_4$ for $k = 3$ and $r = 8$. The representation $\rho^{(3,8)}_4$ is defined as follows:

$$\rho^{(3,8)}_4(\sigma_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
\[
\rho_4^{(3,8)}(\sigma_2) = \begin{pmatrix}
A & 0 & 0 & 0 \\
0 & q & 0 & 0 \\
0 & -1 & 0 & A \\
0 & 0 & A & q
\end{pmatrix},
\]

and
\[
\rho_4^{(3,8)}(\sigma_3) = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & B & 0 & 0 \\
0 & B & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

Here the matrices \(A\) and \(B\) are given by
\[
A = \begin{pmatrix}
q - \frac{1+q}{2+\sqrt{2}} & -1 + q & \sqrt{\frac{1}{\sqrt{2}}(1 - \frac{1}{\sqrt{2}})} \\
-1 + q & q - \frac{1+q}{\sqrt{2}} & \sqrt{\frac{1}{\sqrt{2}}(1 - \frac{1}{\sqrt{2}})} \\
0 & 0 & 0
\end{pmatrix},
\]

and
\[
B = \begin{pmatrix}
q - (1+q)\sqrt{\frac{2}{2+\sqrt{2}}(1 - \frac{2}{2+\sqrt{2}})} & -1 + q & \sqrt{\frac{2(1+q)}{2+\sqrt{2}}(1 - \frac{2}{2+\sqrt{2}})} \\
-1 + q & q - \frac{2(1+q)}{2+\sqrt{2}} & \sqrt{\frac{2}{2+\sqrt{2}}(1 - \frac{2}{2+\sqrt{2}})}
\end{pmatrix}.
\]

5. Faithfulness of \(\rho_4^{(k,r)} : B_4 \to GL_9(\mathbb{C}^*)\)

For each partition of \(n = 4\), we get a matrix whose size depends on the number of the standard tableaux corresponding to the considered partition. The direct sum of the obtained matrices is the \(9 \times 9\) matrix which is the image of the generator \(\sigma_i\) \((i = 1, 2, 3)\).

We get two results concerning Jones-Wenzl representation \(B_4 \to GL_9(\mathbb{C}^*)\). The first one is Theorem 2, which excludes a family of words from belonging to the kernel of the representation. The other result is Theorem 3, which finds a condition under which Jones-Wenzl representation is not faithful.

**Lemma 2.** For any \((k, r)\)-admissible Young diagram of shape \(\lambda = [\lambda_1, \lambda_2, ..., \lambda_k] = [4, 0, ..., 0]\), \(\rho_\lambda^{(k,r)}(\sigma_i) = q\) where \(q = e^{\pm \frac{2\pi i}{r}}\) \((i = 1, 2, 3)\).
Proof. We assign the integers \( \{1, 2, 3, 4\} \) into the boxes of the considered Young diagram of shape \( \lambda = [4, 0, ..., 0] \) so that the entries in the row are increasing.

Let \( t \) be the obtained standard tableau of shape \( \lambda = [4, 0, ..., 0] \). Then, \( t(i) \) is the standard tableau obtained from \( t \) by deleting boxes with entries \( 4, 3, ..., 4 - i + 1 \), where \( i = 1, 2, 3 \).

Since \( r - k \geq 4 \), it follows that \( \lambda_1 - \lambda_k \leq r - k \). Thus, each tableau \( t(i) \) is a \((k, r)\)-admissible Young diagram. This implies that the standard tableau \( t \) is a \((k, r)\)-admissible standard tableau of shape \( \lambda = [4, 0, ..., 0] \).

Let \( s_i(t) \) be the tableau obtained from \( t \) by interchanging the entries \( i \) and \( i + 1 \). Clearly, the entries \( \{1, 2, 3, 4\} \) in \( s_i(t) \) are not increasing. Therefore, \( s_i(t) \) are not standard tableaux for \( i = 1, 2, 3 \) and so they are not \((k, r)\)-admissible.

Consequently, we set \( \beta_{t,i} = 0 \) in \( \pi^{(k,r)}(e_i)(v_t) = \alpha_{t,i}v_t + \beta_{t,i}v_{s_i(t)} \), where \( \{v_t\} \) is the basis of the complex vector space \( V^{(k,r)}_\lambda \).

We have \( \alpha_{t,i} = \frac{[d_{t,i,i+1}]}{[d_{t,i,i+1}]} \) if \( [d_{t,i,i+1}] \neq 0 \). Here \( d_{t,i,m_1,m_2} = (c_1 - c_2) - (r_1 - r_2) \) where \( m_i \) is the entry that appears in row \( r_i \) and column \( c_i \) of the standard tableau \( t \). By simple computations, we get \( d_{t,i,i+1} = -1 \) for \( i = 1, 2, 3 \). This implies that \( \alpha_{t,i} = 0 \). Thus, \( \pi^{(k,r)}(e_i) = 0 \).

By substituting \( \pi^{(k,r)}(e_i) \) in \( \rho^{(k,r)}(\sigma_i) = q - (1 + q)\pi^{(k,r)}(e_i) \), we conclude that \( \rho^{(k,r)}(\sigma_i) = q \). \( \square \)

Theorem 2. Given a word \( u \) of the form \( \sigma_{\alpha_1}^{\beta_1}...\sigma_{\alpha_n}^{\beta_n} \), where \( \alpha_i = 1, 2, 3, \beta_i \in \mathbb{Z} \). Let \( \beta_a \) and \( \beta_b \) be the sums of the exponents of \( (\sigma_{\alpha_i})'s \) and \( (\sigma_{\alpha_i}^{-1})'s \) respectively. If \( \beta_a - \beta_b \neq rw, w \in \mathbb{Z} \), then the words \( u \) don’t belong to the kernel of Jones-Wenzl representation \( \rho_{\lambda}^{(k,r)} \) for all pairs of integers \( (k, r) \).

Proof. We consider the \((k, r)\)-admissible Young diagram of shape \( \lambda = [4, 0, ..., 0] \). Assume, for contradiction, that the words of the form \( \sigma_{\alpha_1}^{\beta_1}...\sigma_{\alpha_n}^{\beta_n} \) belong to the kernel of \( \rho_{\lambda}^{(k,r)} \).

By Lemma 2, \( \rho^{(k,r)}_\lambda(\sigma_i) \) is a \( 1 \times 1 \) block matrix \( (q) \) for \( i = 1, 2, 3 \), and consequently, \( \rho^{(k,r)}_\lambda(\sigma_i^{-1}) \) is a \( 1 \times 1 \) block matrix \( (1/q) \) for \( i = 1, 2, 3 \).

Thus, \( \rho_{\lambda}^{(k,r)}(\sigma_{\alpha_1}^{\beta_1}...\sigma_{\alpha_n}^{\beta_n}) \) has a \( 1 \times 1 \) block matrix of entry \( e^{\pm \frac{\beta_a - \beta_b}{r}} = q^{\beta_a - \beta_b} \) on its diagonal where \( \beta_a \) and \( \beta_b \) are the sums of the exponents of \( (\sigma_i)'s \) and \( (\sigma_i^{-1})'s \) respectively.

We have \( q^{\beta_a - \beta_b} = (e^{\pm \frac{2\pi i}{r}})^{\beta_a - \beta_b} = 1 \). Thus, \( \beta_a - \beta_b = rw, w \in \mathbb{Z} \), a contradiction. \( \square \)
We approach the question of the faithfulness of Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group $B_4$. First, we prove the following lemma:

**Lemma 3.** For $n = 4$ and for each $(k,r)$-admissible standard tableau $t$, the tableaux $s_1(t)$ are not $(k,r)$-admissible standard tableaux.

**Proof.** Let $\lambda$ be a Young diagram of a partition of $n = 4$ such that $\lambda = [\lambda_1, ..., \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$. $\sum_{i=1}^{k} \lambda_i = 4$. We consider all the possible cases.

1. For $\lambda = [3, 1, 0, ..., 0]$, we have 3 standard tableaux $t$ of shape $\lambda$ in which the entries 1 and 2 are either in the same row or in the same column.

2. For $\lambda = [2, 2, 0, ..., 0]$, we have 2 standard tableaux $t$ of shape $\lambda$ in which the entries 1 and 2 are either in the same row or in the same column.

3. For $\lambda = [2, 1, 1, 0, ..., 0]$, we have 3 standard tableaux $t$ of shape $\lambda$ in which the entries 1 and 2 are either in the same row or in the same column.

4. For $\lambda = [4, 0, ..., 0]$, we have 1 standard tableau $t$ of shape $\lambda$ in which the entries 1 and 2 are in the same row.

5. For $\lambda = [1, 1, 1, 1, 0, ..., 0]$, we have 1 standard tableau $t$ of shape $\lambda$ in which the entries 1 and 2 are in the same column.

In all the above cases, we consider the $(k,r)$-admissible standard tableaux $t$ of shape $\lambda$. By interchanging the entries 1 and 2 in each $t$, we obtain the tableau $s_1(t)$. Clearly, the entries in $s_1(t)$ are not in increasing order. Consequently, $s_1(t)$ are not standard tableaux. \[ \square \]

Using Lemma 3, we get the following theorem:

**Theorem 3.** For all integers $k$ and for each even integer $r$, Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group $B_4$ is not faithful.

**Proof.** By Lemma 3, and for all $(k,r)$-admissible standard tableaux $t$, $s_1(t)$ are not $(k,r)$-admissible standard tableaux. Consequently, we set $\beta_{t,1} = 0$ in $\pi_\lambda^{(k,r)}(e_1)(v_t) = \alpha_{t,1}v_t + \beta_{t,1}v_{s_1(t)}$, where $\{v_t\}$ is the basis of the complex vector space $V_\lambda^{(k,r)}$. 


We have $\alpha_{t,1} = \frac{d_{t,1,2} + 1}{2 \cdot |d_{t,1,2}|}$ if $|d_{t,1,2}| \neq 0$. Here $d_{t,1,2} = (c_1 - c_2) - (r_1 - r_2)$, where 1 appears in row $r_1$ and column $c_1$, 2 appears in row $r_2$ and column $c_2$ of each standard tableau $t$.

By simple computations, we get $d_{t,1,2} = -1$ or 1. This implies that $\alpha_{t,1} = 0$ or 1.

By substituting $\pi^{(k,r)}_\lambda(e_1)$ in $\rho^{(k,r)}_{\lambda}(\sigma_1) = q - (1 + q)\pi^{(k,r)}_\lambda(e_1)$, we conclude that $\rho^{(k,r)}_{\lambda}(\sigma_1)$ is either $q$ or $-1$. Therefore, $\rho^{(k,r)}_{4}(\sigma_1)$ is a diagonal matrix having entries equal $q$ or $-1$.

If $r$ is an even integer, then $\rho^{(k,r)}_{4}(\sigma_1^r)$ is an identity matrix. This implies that for any even integer $r$, the word having the form $\sigma_1^r$ belongs to the kernel of Jones-Wenzl representation $\rho^{(k,r)}_{4}$ of the braid group $B_4$. Consequently, the considered representation is not faithful. \hfill \qed

References


