GENERALIZED CANAVATI TYPE $g$-FRACTIONAL POLYA TYPE INEQUALITIES

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Abstract: We present here generalized Canavati type $g$-fractional Polya type inequalities. We cover also the iterated case. Our inequalities are with respect to all $L_p$ norms: $1 \leq p \leq \infty$. We finish with applications.

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1. Introduction

We are motivated by the following famous Polya’s integral inequality, see [11], [12, p. 62], [13] and [14, p. 83].

**Theorem 1.** Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) \, dx.$$  

(1)

We need the following fractional calculus background:

Let $\alpha > 0$, $m = \lfloor \alpha \rfloor$, $\lfloor \cdot \rfloor$ is the integral part, $\beta = \alpha - m$, $0 < \beta < 1$,
\( f \in C([a, b]), [a, b] \subset \mathbb{R}, x \in [a, b] \). The gamma function \( \Gamma \) is given by \( \Gamma(\alpha) = \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt. \) We define the left Riemann-Liouville integral

\[
(J_{a}^{\text{left}} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) \, dt,
\]

\( a \leq x \leq b \). We define the subspace \( C_{a+}^{\alpha}([a, b]) \) of \( C^{m}([a, b]) \):

\[
C_{a+}^{\alpha}([a, b]) = \left\{ f \in C^{m}([a, b]) : J_{a+}^{\alpha-1} f(\eta) \in C^{1}([a, b]) \right\}.
\]

For \( f \in C_{a+}^{\alpha}([a, b]) \), we define the left generalized \( \alpha \) fractional derivative of \( f \) over \([a, b]\) as

\[
D_{a+}^{\alpha} f := \left( J_{a+}^{\alpha-1} f(\eta) \right)',
\]

see [1], p. 24. Canavati first in [6] introduced the above over \([0, 1]\).

Notice that \( D_{a+}^{\alpha} f \in C([a, b]) \).

Furthermore we need:

Let again \( \alpha > 0 \), \( m = \lfloor \alpha \rfloor \), \( \beta = \alpha - m \), \( f \in C([a, b]) \), call the right Riemann-Liouville fractional integral operator by

\[
(J_{b-}^{\text{right}} f)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t) \, dt,
\]

\( x \in [a, b] \), see also [7], [8], [9], [15], [2]. Define the subspace of functions

\[
C_{b-}^{\alpha}([a, b]) = \left\{ f \in C^{m}([a, b]) : J_{b-}^{1-\beta} f(\eta) \in C^{1}([a, b]) \right\}.
\]

Define the right generalized \( \alpha \) fractional derivative of \( f \) over \([a, b]\) as

\[
D_{b-}^{\alpha} f = (-1)^{m-1} \left( J_{b-}^{1-\beta} f(\eta) \right)',
\]

see [2]. We set \( D_{b-}^{0} f = f \). Notice that \( D_{b-}^{\alpha} f \in C([a, b]) \).

We are also motivated by the following fractional Polya type integral inequality without any boundary conditions.

**Theorem 2.** ([4], pp. 1-7) Let \( 0 < \alpha < 1 \), \( f \in C([a, b]) \). Assume \( f \in C_{a+}^{\alpha}([a, a+b]) \) and \( f \in C_{b-}^{\alpha}([\frac{a+b}{2}, b]) \). Set

\[
M(f) = \max \left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, [a, a+b]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty, [\frac{a+b}{2}, b]} \right\}.
\]
Then
\[ \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx \leq M(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}}. \] (9)

Inequality (9) is sharp, namely it is attained by
\[ f_*(x) = \begin{cases} \notag (x-a)^\alpha, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^\alpha, & x \in \left[\frac{a+b}{2}, b\right] \end{cases}, \quad 0 < \alpha < 1. \] (10)

Clearly here non zero constant functions \( f \) are excluded.

In this article we present very general \( g \)-fractional Polya type inequalities.

2. Background

Here we follow [5].

Let \( g : [a, b] \to \mathbb{R} \) be a strictly increasing function. Let \( f \in C^n([a, b]) \), \( n \in \mathbb{N} \). Assume that \( g \in C^1([a, b]) \), and \( g^{-1} \in C^n([g(a), g(b)]) \). Call \( l := f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R} \). It is clear that \( l, l', \ldots, l^{(n)} \) are continuous functions from \([g(a), g(b)]\) into \( f([a, b]) \subseteq \mathbb{R} \).

Let \( \nu \geq 1 \) such that \([\nu] = n, n \in \mathbb{N} \) as above, where \([\cdot]\) is the integral part of the number.

Clearly when \( 0 < \nu < 1 \), \([\nu] = 0 \) and \( n = 0 \). Next we follow [1], pp. 7-9.

I) Let \( h \in C([g(a), g(b)]) \), we define the left Riemann-Liouville fractional integral as
\[ (J_{0}^z \nu h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^z (z-t)^{\nu-1} h(t) \, dt, \] (11)
for \( g(a) \leq z_0 \leq z \leq g(b) \), where \( \Gamma \) is the gamma function; \( \Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1} \, dt \).

We set \( J_{0}^z \nu h = h \).

II) Let \( \alpha := \nu - [\nu] \) \((0 < \alpha < 1)\). We define the subspace \( C_{g(x_0)}^\nu([g(a), g(b)]) \) of \( C^\nu([g(a), g(b)]) \), where \( x_0 \in [a, b] \):
\[ C_{g(x_0)}^\nu([g(a), g(b)]) := \left\{ h \in C^\nu([g(a), g(b)]) : J_{1-a}^{g(x_0)}h([\nu]) \in C^1([g(x_0), g(b)]) \right\}. \] (12)

So let \( h \in C_{g(x_0)}^\nu([g(a), g(b)]) \); we define the left \( g \)-generalized fractional derivative of \( h \) of order \( \nu \), of Canavati type, over \([g(x_0), g(b)]\) as
\[ D_{g(x_0)}^\nu h := \left(J_{1-a}^{g(x_0)}h([\nu])\right)'. \] (13)
Clearly, for $h \in C_{g(x_0)}^\nu ([g(a), g(b)])$, there exists

\[
\left(D_{g(x_0)}^\nu h\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} h^{(\nu)}(t) \, dt,
\]

for all $g(x_0) \leq z \leq g(b)$. In particular, when $f \circ g^{-1} \in C_{g(x_0)}^\nu ([g(a), g(b)])$ we have that

\[
\left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} (f \circ g^{-1})^{(\nu)}(t) \, dt,
\]

for all $g(x_0) \leq z \leq g(b)$. We have

\[
D_{g(x_0)}^n (f \circ g^{-1}) = (f \circ g^{-1})^{(n)} \quad \text{and} \quad D_{g(x_0)}^0 (f \circ g^{-1}) = f \circ g^{-1}.
\]

We mention and need the following left generalized $g$-fractional, of Canavati type, Taylor’s formula:

**Theorem 3.** Let $f \circ g^{-1} \in C_{g(x_0)}^\nu ([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed. 

(i) if $\nu \geq 1$, then

\[
f(x) - f(x_0) = \sum_{k=1}^{\lfloor \nu \rfloor - 1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k
\]

\[
+ \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) \, dt,
\]

all $x \in [a, b] : x \geq x_0$;

(ii) if $0 < \nu < 1$, we get

\[
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) \, dt,
\]

all $x \in [a, b] : x \geq x_0$.

By the change of variables method, see [10], we may rewrite the remainder of (16), (17), as

\[
\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1})\right)(t) \, dt
\]
\[
\frac{1}{\Gamma (\nu)} \int_{x_0}^{x} (g(x) - g(s))^{\nu-1} \left( D_g^{\nu} \left( f \circ g^{-1} \right) \right) (g(s)) \, g'(s) \, ds,
\]
all \( x \in [a, b] : x \geq x_0 \).

We may rewrite (17) as follows: if \( 0 < \nu < 1 \), we have
\[
f(x) = \left( J_{x_0}^{\nu} \left( D_g^{\nu} \left( f \circ g^{-1} \right) \right) \right) (g(x)),
\]
all \( x \in [a, b] : x \geq x_0 \).

II) Next we follow [3], pp. 345-348.

Let \( h \in C ([g(a), g(b)]) \), we define the right Riemann-Liouville fractional integral as
\[
\left( J_{x_0}^{\nu} - h \right) (z) := \frac{1}{\Gamma (\nu)} \int_{z}^{0} (t - z)^{\nu-1} h(t) \, dt,
\]
for \( g(a) \leq z \leq z_0 \leq g(b) \). We set \( J_{x_0}^{0} - h = h \).

Let \( \alpha := \nu - [\nu] \) (0 < \( \alpha < 1 \)). We define the subspace \( C_{g(x_0)}^{\nu} \) \( [g(a), g(b)] \) of \( C^{[\nu]} \) \( [g(a), g(b)] \), where \( x_0 \in [a, b] : \)
\[
C_{g(x_0)}^{\nu} \left( [g(a), g(b)] \right) := \left\{ h \in C^{[\nu]} \left( [g(a), g(b)] \right) : J_{g(x_0)}^{1-\alpha} - h^{[\nu]} \in C^{1} \left( [g(a), g(x_0)] \right) \right\}.
\]
So let \( h \in C_{g(x_0)}^{\nu} \) \( [g(a), g(b)] \); we define the right \( g \)-generalized fractional derivative of \( h \) of order \( \nu \), of Canavati type, over \( [g(a), g(x_0)] \) as
\[
D_{g(x_0)}^{\nu} - h := (-1)^{n-1} \left( J_{g(x_0)}^{1-\alpha} - h^{[\nu]} \right) '.
\]
Clearly, for \( h \in C_{g(x_0)}^{\nu} \) \( [g(a), g(b)] \), there exists
\[
\left( D_{g(x_0)}^{\nu} - h \right) (z) = \frac{(-1)^{n-1}}{\Gamma (1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{[\nu]} (t) \, dt,
\]
for all \( g(a) \leq z \leq g(x_0) \leq g(b) \).

In particular, when \( f \circ g^{-1} \in C_{g(x_0)}^{\nu} \) \( [g(a), g(b)] \) we have that
\[
\left( D_{g(x_0)}^{\nu} \left( f \circ g^{-1} \right) \right) (z) = \frac{(-1)^{n-1}}{\Gamma (1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} \left( f \circ g^{-1} \right)^{[\nu]} (t) \, dt,
\]
for all \( g(a) \leq z \leq g(x_0) \leq g(b) \).
We get that
\[
\left(D^n_{g(x_0)} \left( f \circ g^{-1} \right) \right) (z) = (-1)^n \left( f \circ g^{-1} \right)^{(n)} (z)
\]  
(25)

and \( \left( D^0_{g(x_0)} \left( f \circ g^{-1} \right) \right) (z) = \left( f \circ g^{-1} \right) (z) \), all \( z \in [g(a), g(x_0)] \).

We mention and need the following right generalized \( g \)-fractional, of Canavati type, Taylor’s formula:

**Theorem 4.** Let \( f \circ g^{-1} \in C^\nu_{g(x_0)} ([g(a), g(b)]) \), where \( x_0 \in [a, b] \) is fixed.

(i) if \( \nu \geq 1 \), then
\[
f(x) - f(x_0) = \sum_{k=1}^{[\nu]} \frac{(f \circ g^{-1})^{(k)} (g(x_0))}{k!} (g(x) - g(x_0))^k + \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right)(t) \, dt,
\]  
(26)

all \( a \leq x \leq x_0 \).

(ii) if \( 0 < \nu < 1 \), we get
\[
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right)(t) \, dt,
\]  
(27)

all \( a \leq x \leq x_0 \).

By change of variable, see [10], we may rewrite the remainder of (26), (27), as
\[
\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right)(t) \, dt = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (g(s) - g(x))^{\nu-1} \left(D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right)(g(s)) g'(s) \, ds,
\]  
(28)

all \( a \leq x \leq x_0 \).

We may rewrite (27) as follows:

if \( 0 < \nu < 1 \), we have
\[
f(x) = \left(J^\nu_{g(x_0)} \left(D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right) \right)(g(x)),
\]  
(29)

all \( a \leq x \leq x_0 \leq b \).
III) Denote by
\[ D_{g(x_0)}^{m\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} \ldots D_{g(x_0)}^{\nu} \text{ (m-times), } m \in \mathbb{N}. \] (30)

Also denote by
\[ J_{g(x_0)}^{m\nu} = J_{g(x_0)}^{\nu} J_{g(x_0)}^{\nu} \ldots J_{g(x_0)}^{\nu} \text{ (m-times), } m \in \mathbb{N}. \] (31)

We call \( D_{g(x_0)}^{m\nu} \) an iterated fractional derivative.

We mention and need the following \( g \)-modified and generalized left fractional Taylor’s formula of Canavati type:

**Theorem 5.** Let \( 0 < \nu < 1 \). Assume that \( \left( D_{g(x_0)}^{i\nu} (f \circ g^{-1}) \right) \in C_{g(x_0)}^\nu ([g(a), g(b)]), \) \( x_0 \in [a, b], \) for \( i = 0, 1, \ldots, m. \) Assume also that \( \left( D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) \in C ([g(x_0), g(b)]). \) Then,

\[
f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left( D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right)(z) \, dz \]

\[
= \frac{1}{\Gamma((m + 1)\nu)} \int_{x_0}^{x} (g(x) - g(s))^{(m+1)\nu-1} \times \left( D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right)(g(s)) \, g'(s) \, ds,
\]
nall \( x_0 \leq x \leq b. \)

IV) Denote by
\[ D_{g(x_0)}^{m\nu} - = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} \ldots D_{g(x_0)}^{\nu} \text{ (m-times), } m \in \mathbb{N}. \] (33)

Also denote by
\[ J_{g(x_0)}^{m\nu} - = J_{g(x_0)}^{\nu} J_{g(x_0)}^{\nu} \ldots J_{g(x_0)}^{\nu} \text{ (m-times), } m \in \mathbb{N}. \] (34)

We call \( D_{g(x_0)}^{m\nu} - \) an iterated fractional derivative.

We mention and need the following \( g \)-modified and generalized right fractional Taylor’s formula of Canavati type:
Theorem 6. Let $0 < \nu < 1$. Assume that $D_{g(x_0)}^{\nu}((f \circ g^{-1})) \in C_{g(x_0)}^\nu([g(a), g(b)])$, $x_0 \in [a, b]$, for all $i = 0, 1, \ldots, m$. Assume also that $D_{g(x_0)}^{(m+1)\nu}((f \circ g^{-1})) \in C([g(a), g(x_0)])$. Then,

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left( D_{g(x_0)}^{(m+1)\nu}((f \circ g^{-1})) \right)(z) \, dz$$

$$= \frac{1}{\Gamma((m+1)\nu)} \int_x^{x_0} (g(s) - g(x))^{(m+1)\nu - 1} \times \left( D_{g(x_0)}^{(m+1)\nu}((f \circ g^{-1})) \right)(g(s)) \, g'(s) \, ds,$$

all $a \leq x \leq x_0 \leq b$.

In what follows next it is based on this background.

3. Main results

We present the following generalized $g$-fractional Polya type integral inequalities without any boundary conditions.

Theorem 7. Let $0 < \nu < 1$, $f \in C([a, b])$, $g : [a, b] \to \mathbb{R}$ be strictly increasing and $g \in C^1([a, b])$. Assume

$$f \circ g^{-1} \in C_{g(a)}^\nu \left( \left[ \frac{g(a) + g(b)}{2} \right] \right),$$

and

$$f \circ g^{-1} \in C_{g(b)}^\nu \left( \left[ \frac{g(a) + g(b)}{2}, g(b) \right] \right).$$

Set

$$M(f, g) := \max \left\{ \left\| D_{g(a)}^\nu((f \circ g^{-1})) \right\|_{\infty, [g(a), \frac{g(a) + g(b)}{2}]}, \left\| D_{g(b)}^\nu((f \circ g^{-1})) \right\|_{\infty, [\frac{g(a) + g(b)}{2}, g(b)]} \right\}.$$  \hspace{1cm} (36)

Then

$$\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq M(f, g) \frac{(g(b) - g(a))^{\nu + 1}}{\Gamma(\nu + 2) 2^\nu}. \hspace{1cm} (37)$$

Inequality (37) is sharp, namely it is attained by $f_*$ such that
\[(f_\ast \circ g^{-1}) (z) = \left\{ \begin{align*}
(z-g(a))^\nu, & \; z \in \left[ g(a), \frac{g(a)+g(b)}{2} \right] \\
g(b)-z)^\nu, & \; z \in \left[ g(a), \frac{g(a)+g(b)}{2}, g(b) \right]
\end{align*} \right\}, \; 0 < \nu < 1. \quad (38)\]

Clearly here non zero constant functions \( f \circ g^{-1} \) are excluded.

**Proof.** Notice that

\[
\left| \int_a^b f \, dg \right| = \left| \int_a^b f \, g' \, dx \right| \leq \int_a^b |f| \, g' \, dx = \int_a^b |f| \, dg.
\]

Let \( x_0 \in [a, b] \) be such that \( g(x_0) = \frac{g(a)+g(b)}{2} \), that is \( x_0 = g^{-1} \left( \frac{g(a)+g(b)}{2} \right) \).

Let \( f \circ g^{-1} \in C^\nu_{g(a)} \left( \left[ g(a), \frac{g(a)+g(b)}{2}, g(b) \right] \right), \; 0 < \nu < 1. \) By Theorem 3 we have that

\[
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left( D^\nu_{g(a)} \left( f \circ g^{-1} \right) \right) (t) \, dt,
\]

all \( x \in [a, x_0] \).

Assume also \( f \circ g^{-1} \in C^\nu_{g(b)} \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right), \) by Theorem 4 we have that

\[
f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left( D^\nu_{g(b)} \left( f \circ g^{-1} \right) \right) (t) \, dt,
\]

all \( x \in [x_0, b] \).

By (39) we get

\[
|f(x)| \leq \frac{1}{\Gamma(\nu)} \left( \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \, dt \right) \left\| D^\nu_{g(a)} \left( f \circ g^{-1} \right) \right\|_{\infty, \left[ g(a), \frac{g(a)+g(b)}{2} \right]} \left( g(x) - g(a) \right)^\nu \Gamma(\nu + 1).
\]

That is,

\[
|f(x)| \leq \left\| D^\nu_{g(a)} \left( f \circ g^{-1} \right) \right\|_{\infty, \left[ g(a), \frac{g(a)+g(b)}{2} \right]} \frac{(g(x) - g(a))^\nu}{\Gamma(\nu + 1)}, \quad (41)
\]

for any \( x \in [a, x_0] \).

Similarly, by (40) we get

\[
|f(x)| \leq \frac{1}{\Gamma(\nu)} \left( \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \, dt \right) \left\| D^\nu_{g(b)} \left( f \circ g^{-1} \right) \right\|_{\infty, \left[ \frac{g(a)+g(b)}{2}, g(b) \right]} \frac{(g(x) - g(a))^\nu}{\Gamma(\nu + 1)},
\]

for any \( x \in [x_0, b] \).
Thus, we have
\[
\|f(x)\| \leq \|D_{g(b)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]} \frac{(g(b) - g(x))^\nu}{\Gamma (\nu + 1)},
\]
for any \(x \in [x_0, b]\).

Thus, we have
\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg = \int_a^b |f| \, g' \, dx
\]
\[
= \int_a^{x_0} |f(x)| \, g'(x) \, dx + \int_{x_0}^b |f(x)| \, g'(x) \, dx \quad \text{(by (42), (44))}
\]
\[
\frac{\|D_{g(a)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \int_a^{x_0} (g(x) - g(a))^\nu g'(x) \, dx
\]
\[
+ \frac{\|D_{g(b)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \int_{x_0}^b (g(b) - g(x))^\nu g'(x) \, dx
\]
\[
= \frac{\|D_{g(a)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \frac{(g(x_0) - g(a))^\nu + 1}{(\nu + 1)}
\]
\[
+ \frac{\|D_{g(b)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \frac{(g(b) - g(x_0))^{\nu + 1}}{(\nu + 1)}
\]
\[
= \frac{1}{\Gamma (\nu + 2)} \left[ \frac{\|D_{g(a)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \frac{(g(b) - g(a))^{\nu + 1}}{2^{\nu + 1}} \right]
\]
\[
= \frac{1}{\Gamma (\nu + 2)} \frac{(g(b) - g(a))^{\nu + 1}}{2^{\nu + 1}} \left[ \frac{\|D_{g(a)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \frac{(g(b) - g(a))^{\nu + 1}}{2^{\nu + 1}} \right]
\]
\[
= \frac{1}{\Gamma (\nu + 2)} \frac{2^{\nu + 1}}{2^{\nu + 1}} \left[ \frac{\|D_{g(a)}^\nu (f \circ g^{-1})\|_{\infty, \left[\frac{g(a)+g(b)}{2}, g(b)\right]}}{\Gamma (\nu + 1)} \frac{(g(b) - g(a))^{\nu + 1}}{2^{\nu + 1}} \right]
\]
We have proved inequality (37). Next we prove sharpness of (37).

Here \( \nu = 0 \) and \( \alpha = \nu \). We see that

\[
\left( J^{g(a)}_{1-\alpha} (f_\ast \circ g^{-1}) \right) (z) = \frac{1}{\Gamma (1-\alpha)} \int_z^{g(a)} (z-t)^{-(\nu+1)} \left( t - g(a) \right)^{(\nu+1)-1} dt
\]

(by [16], p. 256)

\[
= \frac{1}{\Gamma (1-\alpha)} \frac{\Gamma (1-\alpha) \Gamma (\nu+1)}{\Gamma (2)} (z - g(a)) = \Gamma (\nu+1) (z - g(a)).
\]

Therefore,

\[
\left( D^{\nu}_{g(a)} (f_\ast \circ g^{-1}) \right) (z) = \Gamma (\nu+1), \text{ for all } z \in \left[ g(a), \frac{g(a) + g(b)}{2} \right].
\]

That is,

\[
\left\| D^{\nu}_{g(a)} (f_\ast \circ g^{-1}) \right\|_\infty, \left[ g(a), \frac{g(a) + g(b)}{2} \right] = \Gamma (\nu+1).
\]

Furthermore we have

\[
- \left( J^{1-\alpha}_{g(b)-} (f_\ast \circ g^{-1}) \right) (z) = \frac{-1}{\Gamma (1-\alpha)} \int_z^{g(b)} (t-z)^{-\alpha} (g(b) - t)^\nu dt
\]

(by [16], p. 256)

\[
= -\frac{1}{\Gamma (1-\alpha)} \int_z^{g(b)} (g(b) - t)^{(\nu+1)-1} (t-z)^{(1-\alpha)-1} dt
\]

for all \( z \in \left[ \frac{g(a) + g(b)}{2}, g(b) \right] \).

Hence

\[
\left( D^{\nu}_{g(b)-} (f_\ast \circ g^{-1}) \right) (z) = \Gamma (\nu+1),
\]

and

\[
\left\| D^{\nu}_{g(b)-} (f_\ast \circ g^{-1}) \right\|_\infty, \left[ \frac{g(a) + g(b)}{2}, g(b) \right] = \Gamma (\nu+1).
\]
Consequently we get that

\[ M(f_*, g) = \Gamma(\nu + 1). \quad (55) \]

Applying \( f_* \) into (37) we obtain:

\[ \text{R.H.S (37) for } f_* = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \frac{(g(b) - g(a))^{\nu + 1}}{2^\nu} = \frac{(g(b) - g(a))^{\nu + 1}}{(\nu + 1) 2^\nu}. \quad (56) \]

We get the same result when we apply \( f_* \) in the

\[ \text{L.H.S (37)} = \left| \int_a^b f_*(x) \, dg(x) \right| \]

\[ = \left| \int_a^{x_0} f_*(x) \, dg(x) + \int_{x_0}^b f_*(x) \, dg(x) \right| \]

\[ = \left| \int_a^{x_0} (f_* \circ g^{-1})(g(x)) \, dg(x) + \int_{x_0}^b (f_* \circ g^{-1})(g(x)) \, dg(x) \right| \]

\[ = \left| \int_{g(a)}^{g(x_0)} (f_* \circ g^{-1})(z) \, dz + \int_{g(x_0)}^{g(b)} (f_* \circ g^{-1})(z) \, dz \right| \]

\[ = \left| \int_{g(a)}^{g(x_0)} (z - g(a))^{\nu} \, dz + \int_{g(x_0)}^{g(b)} (g(b) - z)^{\nu} \, dz \right| \]

\[ = \frac{(g(x_0) - g(a))^{\nu + 1}}{\nu + 1} + \frac{(g(b) - g(x_0))^{\nu + 1}}{\nu + 1} \]

\[ = \frac{2}{\nu + 1} \frac{(g(b) - g(a))^{\nu + 1}}{2^{\nu + 1}} = \frac{(g(b) - g(a))^{\nu + 1}}{(\nu + 1) 2^\nu}. \quad (58) \]

So equality holds in (37) for \( f_* \). Furthermore we notice that

\[ (f_* \circ g^{-1}) \left( \left( \frac{g(a) + g(b)}{2} \right)^+ \right) = (f_* \circ g^{-1}) \left( \left( \frac{g(a) + g(b)}{2} \right)^- \right) \]

\[ = \left( \frac{g(b) - g(a)}{2} \right)^\nu, \quad (59) \]

thus \( f_* \circ g^{-1} \) is continuous on \([g(a), g(b)]\) and hence \( f_* \in C([a, b]) \).

Sharpness of (37) has been proved. The theorem is proved completely. \( \square \)
Remark 8. When $\nu \geq 1$, thus $n = \lceil \nu \rceil \geq 1$, and by assuming that $(f \circ g^{-1})^{(k)}(g(a)) = (f \circ g^{-1})^{(k)}(g(b)) = 0$, for $k = 0, 1, \ldots, n-1$, we can prove inequality (37) again. Here also $f \in C^n([a, b])$ and $g^{-1} \in C^n([g(a), g(b)])$.

The function $(f \circ g^{-1})^{(n)}$ cannot be a constant different than zero, equivalently, $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree $n$.

We continue with a fractional iterated Polya type inequality:

Theorem 9. Let $0 < \nu < 1$, $f \in C([a, b])$, $g : [a, b] \to \mathbb{R}$ be strictly increasing and $g \in C^1([a, b])$. Assume that $\left( D_{g(a)}^{\nu}(f \circ g^{-1}) \right) \in C^\nu_g(a, g^{-1}(a))$, for $i = 0, 1, \ldots, m \in \mathbb{N}$, and $\left( D_{g(b)}^{(m+1)\nu}(f \circ g^{-1}) \right) \in C^\nu_g(b, g^{-1}(b))$. Also assume that

\[
\left( D_{g(a)}^{\nu}(f \circ g^{-1}) \right) \in C^\nu_g(a, g^{-1}(a))
\]

for $i = 0, 1, \ldots, m \in \mathbb{N}$, and

\[
\left( D_{g(b)}^{(m+1)\nu}(f \circ g^{-1}) \right) \in C^\nu_g(b, g^{-1}(b))
\]

Set

\[
M^*(f, g) := \max \left\{ \left\| D_{g(a)}^{(m+1)\nu}(f \circ g^{-1}) \right\|_{\infty, \frac{g(a) + g(b)}{2}}, \left\| D_{g(b)}^{(m+1)\nu}(f \circ g^{-1}) \right\|_{\infty, \frac{g(a) + g(b)}{2}} \right\}.
\]

Then

\[
\int_a^b f \, dg \leq \int_a^b |f| \, dg \leq M^*(f, g) \frac{(g(b) - g(a))^{(m+1)\nu+1}}{\Gamma((m + 1)\nu + 2) 2^{(m+1)\nu}}.
\]

Proof. By Theorem 5 we have

\[
f(x) = \frac{1}{\Gamma((m + 1)\nu)} \int_{g(a)}^{g(x)} \frac{(g(z) - z)^{(m+1)\nu-1}}{(m + 1)\nu} \, dz.
\]

\[
\times \left( D_{g(a)}^{(m+1)\nu}(f \circ g^{-1}) \right) (z) \, dz.
\]
all \( x \in [a, x_0] \), where \( x_0 := g^{-1}\left(\frac{g(a) + g(b)}{2}\right) \).

By Theorem 6 we have
\[
f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(b)} (z - g(x))^{(m+1)\nu-1}dz \\
\times \left( D_{g(b)}^{(m+1)\nu} (f \circ g^{-1}) \right)(z) dz,
\]
(63)

all \( x \in [x_0, b] \).

By (62) we get
\[
|f(x)| \leq \frac{1}{\Gamma((m+1)\nu)} \\
\left( \int_{g(a)}^{g(x)} (g(x) - z)^{(m+1)\nu-1}dz \right) \left| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right|_{\infty, [g(a), g(a) + g(b)]}
\]
(64)

for all \( x \in [a, x_0] \).

Similarly, by (63), we obtain
\[
|f(x)| \leq \frac{1}{\Gamma((m+1)\nu)} \\
\left( \int_{g(x)}^{g(b)} (z - g(x))^{(m+1)\nu-1}dz \right) \left| D_{g(b)}^{(m+1)\nu} (f \circ g^{-1}) \right|_{\infty, [g(a) + g(b), g(b)]}
\]
(65)

for all \( x \in [x_0, b] \).

Thus, we have
\[
\left| \int_{a}^{b} f dg \right| \leq \int_{a}^{b} |f|dg = \int_{a}^{b} |f|g' dx \\
\leq \int_{a}^{x_0} |f(x)|g'(x)dx + \int_{x_0}^{b} |f(x)|g'(x)dx \text{ (by (64), (65))}
\]
(66)

\[
+ \frac{\left| D_{g(b)}^{(m+1)\nu} (f \circ g^{-1}) \right|_{\infty, [g(a) + g(b), g(b)]}}{\Gamma((m+1)\nu+1)} \int_{a}^{x_0} (g(x) - g(a))^{(m+1)\nu}g'(x)dx
\]

\[
+ \frac{\left| D_{g(b)}^{(m+1)\nu} (f \circ g^{-1}) \right|_{\infty, [g(a) + g(b), g(b)]}}{\Gamma((m+1)\nu+1)} \int_{x_0}^{b} (g(b) - g(x))^{(m+1)\nu}g'(x)dx
\]
be a strictly increasing function. Assume that \( C_{\|} \| \) for \( k \geq 0 \), for \( \nu \geq 1, n = [\nu] \), and \( f \in C^n ([a, b]) \), \( g : [a, b] \to \mathbb{R} \) be a strictly increasing function. Assume that \( g \in C^1 ([a, b]) \) and \( g^{-1} \in C^n ([g(a), g(b)]) \). Also assume \( f \circ g^{-1} \in C^\nu_{g(a)} \left( [g(a), \frac{g(a)+g(b)}{2}] \right) \), and \( f \circ g^{-1} \in C^\nu_{g(b)} \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right) \), and that \( (f \circ g^{-1})^{(k)} (g(a)) = \frac{(f \circ g^{-1})^{(k)}}{2^k} (g(b)) = 0 \), for \( k = 0, 1, \ldots, n - 1 \). Set

\[
M_1 (f, g) := \max \left\{ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_1 ([g(a), \frac{g(a)+g(b)}{2}])}, \left\| D^\nu_{g(b)} (f \circ g^{-1}) \right\|_{L_1 ([\frac{g(a)+g(b)}{2}, g(b)])} \right\}.
\]

Then

\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq M_1 (f, g) \frac{(g(b) - g(a))^{\nu}}{\Gamma (\nu + 1) 2^{\nu-1}}.
\]

Here \( f \circ g^{-1} \) cannot be a non-trivial polynomial of degree \( n \).

Next we give an \( L_1 \) generalized \( g \)-fractional Polya inequality:

**Theorem 10.** Let \( \nu \geq 1, n = [\nu] \), and \( f \in C^n ([a, b]) \), \( g : [a, b] \to \mathbb{R} \) be a strictly increasing function. Assume that \( g \in C^1 ([a, b]) \) and \( g^{-1} \in C^n ([g(a), g(b)]) \). Also assume \( f \circ g^{-1} \in C^\nu_{g(a)} \left( [g(a), \frac{g(a)+g(b)}{2}] \right) \), and \( f \circ g^{-1} \in C^\nu_{g(b)} \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right) \), and that \( (f \circ g^{-1})^{(k)} (g(a)) = \frac{(f \circ g^{-1})^{(k)}}{2^k} (g(b)) = 0 \), for \( k = 0, 1, \ldots, n - 1 \). Set

\[
M_1 (f, g) := \max \left\{ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_1 ([g(a), \frac{g(a)+g(b)}{2}])}, \left\| D^\nu_{g(b)} (f \circ g^{-1}) \right\|_{L_1 ([\frac{g(a)+g(b)}{2}, g(b)])} \right\}.
\]

Then

\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq M_1 (f, g) \frac{(g(b) - g(a))^{\nu}}{\Gamma (\nu + 1) 2^{\nu-1}}.
\]
Proof. Let \( f \circ g^{-1} \in C_{g(a)}^{\nu} \left( \left[ g(a), \frac{g(a)+g(b)}{2} \right] \right) \), \( \nu \geq 1 \), and \( (f \circ g^{-1})^{(k)}(g(a)) = 0 \), for \( k = 0, 1, \ldots, n-1 \), \( n = \lceil \nu \rceil \). By Theorem 3 we have that

\[
 f(x) = \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left( D_{g(a)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \, dt, \tag{71}
\]

all \( x \in [a, x_0] \), where \( x_0 := g^{-1}\left( \frac{g(a)+g(b)}{2} \right) \).

Let \( f \circ g^{-1} \in C_{g(b)}^{\nu} \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right) \), and \( (f \circ g^{-1})^{(k)}(g(b)) = 0 \), for \( k = 0, 1, \ldots, n-1 \). By Theorem 4 we have that

\[
 f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left( D_{g(b)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \, dt, \tag{72}
\]

all \( x \in [x_0, b] \).

By (71) we have

\[
 |f(x)| \leq \frac{1}{\Gamma(\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu-1} \left| \left( D_{g(a)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \right| \, dt \leq \frac{(g(x) - g(a))^{\nu-1}}{\Gamma(\nu)} \int_{g(a)}^{g(x)} \left| \left( D_{g(a)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \right| \, dt, \tag{73}
\]

for all \( x \in [a, x_0] \).

Similarly, by (72) we get

\[
 |f(x)| \leq \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu-1} \left| \left( D_{g(b)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \right| \, dt \leq \frac{(g(b) - g(x))^{\nu-1}}{\Gamma(\nu)} \int_{g(x)}^{g(b)} \left| \left( D_{g(b)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \right| \, dt \leq \frac{(g(b) - g(x))^{\nu-1}}{\Gamma(\nu)} \left\| D_{g(b)}^{\nu} \left( f \circ g^{-1} \right) \right\|_{L_1 \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right)} , \tag{74}
\]

for all \( x \in [x_0, b] \).

Thus, we have

\[
 \left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg = \int_a^b |f| \cdot g' \, dx = \int_a^{x_0} |f(x)| \cdot g'(x) \, dx + \int_{x_0}^b |f(x)| \cdot g'(x) \, dx \overset{\text{(by (73), (74))}}{\leq}
\]
Then:

\[
\frac{\left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_1\left(\left[ g(a), \frac{g(a)+g(b)}{2} \right] \right)}}{\Gamma (\nu)} \int_{a}^{x_0} (g (x) - g (a))^{\nu-1} g' (x) \, dx
\]

\[
+ \frac{\left\| D^\nu_{g(b)} - (f \circ g^{-1})\right\|_{L_1\left(\left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right)}}{\Gamma (\nu)} \int_{x_0}^{b} (g (b) - g (x))^{\nu-1} g' (x) \, dx
\]

\[
= \frac{\left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_1\left(\left[ g(a), \frac{g(a)+g(b)}{2} \right] \right)}}{\Gamma (\nu + 1)} (g(x_0) - g(a))^{\nu}
\]

\[
+ \frac{\left\| D^\nu_{g(b)} - (f \circ g^{-1})\right\|_{L_1\left(\left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right)}}{\Gamma (\nu + 1)} (g(b) - g(x_0))^{\nu}
\]

\[
\leq \frac{1}{\Gamma (\nu + 1)} \frac{(g(b) - g(a))^{\nu}}{2^{\nu-1}}
\]

proving the claim.

We continue with a $L_q$ generalized $g$-fractional Polya type inequality:

**Theorem 11.** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$, $f \in C^n ([a, b])$, $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. Assume that $g \in C^1 ([a, b])$ and $g^{-1} \in C^n ([g(a), g(b)])$. Also assume $f \circ g^{-1} \in C^\nu_{g(a)} \left( \left[ g(a), \frac{g(a)+g(b)}{2} \right] \right)$, and $f \circ g^{-1} \in C^\nu_{g(b)} \left( \left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right)$, and that $(f \circ g^{-1})^{(k)} (g(a)) = (f \circ g^{-1})^{(k)} (g(b)) = 0$, for $k = 0, 1, ..., n - 1$. When $\frac{1}{q} < \nu < 1$, the last boundary conditions are void. Set

\[
M_q (f, g) := \max \left\{ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_q\left(\left[ g(a), \frac{g(a)+g(b)}{2} \right] \right)}, \right\}
\]

\[
\left\| D^\nu_{g(b)} - (f \circ g^{-1})\right\|_{L_q\left(\left[ \frac{g(a)+g(b)}{2}, g(b) \right] \right)} \right\}.
\]

Then
\[ \left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq \frac{M_q(f; g)}{2^{\nu - \frac{1}{q}} \Gamma (\nu) \left(p (\nu - 1) + 1\right)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (g(b) - g(a))^\nu + \frac{1}{p}. \] (78)

Again here \( f \circ g^{-1} \) cannot be a non-trivial polynomial of degree \( n \).

**Proof.** Let \( f \circ g^{-1} \in C_{g(a)}^\nu \left( \left[ g(a), \frac{g(a) + g(b)}{2} \right] \right), \ \nu > 0 \), and \( (f \circ g^{-1})^{(k)}(g(a)) = 0, \) for \( k = 0, 1, \ldots, n - 1, \ n = \lceil \nu \rceil \). By Theorem 3 we have that

\[ f(x) = \frac{1}{\Gamma (\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^\nu - 1 \left( D_{g(a)}^\nu (f \circ g^{-1}) \right) (t) \, dt, \] (79)

all \( x \in [a, x_0] \), where \( x_0 := g^{-1} \left( \frac{g(a) + g(b)}{2} \right) \).

Let \( f \circ g^{-1} \in C_{g(b)-}^\nu \left( \left[ \frac{g(a) + g(b)}{2}, g(b) \right] \right) \), and \( (f \circ g^{-1})^{(k)}(g(b)) = 0, \) for \( k = 0, 1, \ldots, n - 1. \) By Theorem 4 we have that

\[ f(x) = \frac{1}{\Gamma (\nu)} \int_{g(x)}^{g(b)} (t - g(x))^\nu - 1 \left( D_{g(b)-}^\nu (f \circ g^{-1}) \right) (t) \, dt, \] (80)

all \( x \in [x_0, b] \).

By (79) we have

\[
\frac{1}{\Gamma (\nu)} \left( \int_{g(a)}^{g(x)} (g(x) - t)^{\nu - 1} \left| \left( D_{g(a)}^\nu (f \circ g^{-1}) \right) (t) \right| \, dt \right) \leq \frac{1}{\Gamma (\nu)} \int_{g(a)}^{g(x)} (g(x) - t)^{\nu - 1} \left( \left| \left( D_{g(a)}^\nu (f \circ g^{-1}) \right) (t) \right|^q \, dt \right) \frac{1}{q} \frac{1}{p}
\]

\[
\leq \frac{1}{\Gamma (\nu)} \frac{(g(x) - g(a))^{\nu + \frac{1}{p}}}{(p (\nu - 1) + 1)^{\frac{1}{p}}} \left\| D_{g(a)}^\nu (f \circ g^{-1}) \right\|_{L_q \left( [g(a), \frac{g(a) + g(b)}{2}] \right)}, \] (81)

for all \( x \in [a, x_0] \).

Similarly, by (80) we get

\[
\frac{1}{\Gamma (\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu - 1} \left| \left( D_{g(b)-}^\nu (f \circ g^{-1}) \right) (t) \right| \, dt \leq \frac{1}{\Gamma (\nu)} \int_{g(x)}^{g(b)} (t - g(x))^{\nu - 1} \left( \left| \left( D_{g(b)-}^\nu (f \circ g^{-1}) \right) (t) \right|^q \, dt \right) \frac{1}{q} \frac{1}{p}
\]

\[
\leq \frac{1}{\Gamma (\nu)} \frac{(g(b) - g(x))^{\nu + \frac{1}{p}}}{(p (\nu - 1) + 1)^{\frac{1}{p}}} \left\| D_{g(b)-}^\nu (f \circ g^{-1}) \right\|_{L_q \left( [g(a), \frac{g(a) + g(b)}{2}] \right)}. \]
proving the claim. Thus, we have

\[ \int_a^b f \, dg \leq \int_a^b |f| \, dg = \]

\[ \int_a^b |f(x)| \, g'(x) \, dx + \int_{x_0}^a |f(x)| \, g'(x) \, dx \leq \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \]

\[ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_q\left([g(a), \frac{g(a)+g(b)}{2}, g(b)]\right)} \int_{x_0}^a (g(x) - g(a))^{\nu-1+\frac{1}{p}} g'(x) \, dx \]

\[ + \left\| D^\nu_{g(b)} (f \circ g^{-1}) \right\|_{L_q\left([\frac{g(a)+g(b)}{2}, g(b)]\right)} \int_{x_0}^b (g(b) - g(x))^{\nu-1+\frac{1}{p}} g'(x) \, dx \]

\[ = \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\nu + \frac{1}{p}\right) \]

\[ \times \left[ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_q\left([g(a), g(a)+g(b)]\right)} (g(x_0) - g(a))^{\nu+\frac{1}{p}} \right. \]

\[ + \left. \left\| D^\nu_{g(b)} (f \circ g^{-1}) \right\|_{L_q\left([\frac{g(a)+g(b)}{2}, g(b)]\right)} (g(b) - g(x_0))^{\nu+\frac{1}{p}} \right] \]

\[ = \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\nu + \frac{1}{p}\right) \]

\[ \frac{(g(b) - g(a))^{\nu+\frac{1}{p}}}{2^{\nu+\frac{1}{p}}} \times \]

\[ \left[ \left\| D^\nu_{g(a)} (f \circ g^{-1}) \right\|_{L_q\left([g(a), \frac{g(a)+g(b)}{2}], g(b)]\right)} + \left\| D^\nu_{g(b)} (f \circ g^{-1}) \right\|_{L_q\left([\frac{g(a)+g(b)}{2}, g(b)]\right)} \right] \]

\[ \leq \frac{M_q(f, g)}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\nu + \frac{1}{p}\right) \frac{(g(b) - g(a))^{\nu+\frac{1}{p}}}{2^{\nu+\frac{1}{p}}}, \]

proving the claim. \[\square\]

Combining facts from Theorems 7, 10, 11 and Remark 8, we get:
\textbf{Theorem 12.} Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu \geq 1$, $n = \lceil \nu \rceil$, $f \in C^n ([a, b])$, $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. Assume that $g \in C^1 ([a, b])$ and $g^{-1} \in C^n ([g(a), g(b)])$. Also assume $f \circ g^{-1} \in C^\nu_{g(a)} \left( \left[ g(a), \frac{g(a) + g(b)}{2} \right] \right)$, and $f \circ g^{-1} \in C^\nu_{g(b)} \left( \left[ \frac{g(a) + g(b)}{2}, g(b) \right] \right)$, and that

\[
(f \circ g^{-1})^{(k)} (g(a)) = (f \circ g^{-1})^{(k)} (g(b)) = 0,
\]

for $k = 0, 1, \ldots, n - 1$. Then

\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq \min \left\{ M(f, g) \frac{(g(b) - g(a))^{\nu + 1}}{\Gamma(\nu + 2) 2^\nu}, M_1(f, g) \frac{(g(b) - g(a))^\nu}{\Gamma(\nu + 1) 2^{\nu - 1}}, M_q(f, g) \frac{2^{\nu - \frac{1}{2}} \Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right) (g(b) - g(a))^{\nu + \frac{1}{p}}}{\Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right) (g(b) - g(a))^{\nu + \frac{1}{p}}} \right\},
\]

where $M(f, g)$ as in (36), $M_1(f, g)$ as in (69) and $M_q(f, g)$ as in (77). Above $f \circ g^{-1}$ cannot be a non-trivial polynomial of degree $n$.

\textbf{Corollary 13.} Here all as in Theorem 12. Then

\[
\left| \frac{1}{g(b) - g(a)} \int_a^b f \, dg \right| \leq \frac{1}{g(b) - g(a)} \int_a^b |f| \, dg \leq \min \left\{ M(f, g) \frac{(g(b) - g(a))^{\nu + 1}}{\Gamma(\nu + 2) 2^\nu}, M_1(f, g) \frac{(g(b) - g(a))^\nu}{\Gamma(\nu + 1) 2^{\nu - 1}}, M_q(f, g) \frac{2^{\nu - \frac{1}{2}} \Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right) (g(b) - g(a))^{\nu + \frac{1}{p}}}{\Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right) (g(b) - g(a))^{\nu + \frac{1}{p}}} \right\}.
\]

We continue with an $L_1$ iterated fractional Polya type inequality:

\textbf{Theorem 14.} All as in Theorem 9, with $\frac{1}{m+1} \leq \nu < 1$. Set

\[
M_1^* (f, g) := \max \left\{ \left\| D^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_1(\left[ g(a), \frac{g(a) + g(b)}{2} \right])} \right\},
\]

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\[ \left\| D_{g(b)}^{\nu} (f \circ g^{-1}) \right\|_{L_1\left(\frac{g(a)+g(b)}{2},g(b)\right)} \right\}. \] (87)

Then
\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq M_q^* (f, g) \frac{(g(b) - g(a))^{(m+1)\nu}}{\Gamma((m+1)\nu + 1) 2^{(m+1)\nu - 1}}. \] (88)

**Proof.** Similar to Theorem 10. \(\Box\)

We continue with an \(L_p\) iterated fractional Polya type inequality:

**Theorem 15.** All as in Theorem 9. Let \(p, q > 1: \frac{1}{p} + \frac{1}{q} = 1\), such that \(\frac{1}{(m+1)q} < \nu < 1\). Set
\[
M_q^* (f, g) := \max \left\{ \left\| D_{g(a)}^{(m+1)\nu} (f \circ g^{-1}) \right\|_{L_q\left(\frac{g(a)+g(b)}{2},g(b)\right)}, \right\}. \] (89)

Then
\[
\left| \int_a^b f \, dg \right| \leq \int_a^b |f| \, dg \leq M_q^* (f, g) \frac{(g(b) - g(a))^{(m+1)\nu+\frac{1}{p}}}{2^{(m+1)\nu - \frac{1}{q}} \Gamma ((m+1)\nu) (p ((m+1)\nu - 1) + 1) \frac{1}{q} ((m+1)\nu + \frac{1}{p})}. \] (90)

**Proof.** Similar to Theorem 11. \(\Box\)

Applications follow:

**Proposition 16.** Let \(0 < \nu < 1\), \(f \in C([a,b])\). Assume \(f \circ \ln x \in C^\nu_{e^a} \left([e^a, e^{\frac{a+b}{2}}]\right)\), and \(f \circ \ln x \in C^\nu_{e^{-b}} \left([e^{\frac{a+b}{2}}, e^b]\right)\). Set
\[
M(f, e^x) := \max \left\{ \left\| D_{e^a}^{\nu} (f \circ \ln) \right\|_{\infty, [e^a, e^{\frac{a+b}{2}}]}, \right\}, \] (91)

Then
\[
\left| \int_a^b f(x) e^x \, dx \right| \leq \int_a^b |f(x)| e^x \, dx \leq M(f, e^x) \frac{(e^b - e^a)^{\nu+1}}{\Gamma(\nu+2) 2^\nu}. \] (92)
Inequality (92) is sharp, namely it is attained by \( f^* \) such that

\[
(f^* \circ \ln x)(z) = \begin{cases} 
(z - e^a)^\nu, & z \in \left[ e^a, \frac{e^a + e^b}{2} \right] \\
(e^b - z)^\nu, & z \in \left[ \frac{e^a + e^b}{2}, e^b \right]
\end{cases}, \quad 0 < \nu < 1.
\]

Clearly here non zero constant functions \( f \circ \ln x \) are excluded.

Proof. By Theorem 7.

We continue with

**Proposition 17.** Here \([a, b] \subset (0, +\infty), 0 < \nu < 1, f \in C([a, b])\). Assume that \((D_{lna}^\nu (f \circ e^x)) \in C_{lna}^\nu \left( \left[ \ln a, \frac{\ln(ab)}{2} \right] \right)\), for \( i = 0, 1, ..., m \in \mathbb{N} \), and \((D_{lna}^{(m+1)\nu} (f \circ e^x)) \in C \left( \left[ \ln a, \frac{\ln(ab)}{2} \right] \right)\). Also assume that \((D_{lnb-}^\nu (f \circ e^x)) \in C_{lnb-}^\nu \left( \left[ \frac{\ln(ab)}{2}, \ln b \right] \right)\), for \( i = 0, 1, ..., m \in \mathbb{N} \), and \((D_{lnb-}^{(m+1)\nu} (f \circ e^x)) \in C \left( \left[ \frac{\ln(ab)}{2}, \ln b \right] \right)\). Set

\[
M^* (f, \ln x) := \max \left\{ \left\| D_{lna}^{(m+1)\nu} (f \circ e^x) \right\|_{\infty, \left[ \ln a, \frac{\ln(ab)}{2} \right]}, \left\| D_{lnb-}^{(m+1)\nu} (f \circ e^x) \right\|_{\infty, \left[ \frac{\ln(ab)}{2}, \ln b \right]} \right\}.
\]

Then

\[
\left| \int_a^b \frac{f(x)}{x} dx \right| \leq \int_a^b \left| \frac{f(x)}{x} \right| dx \leq M^* (f, \ln x) \frac{\left( \ln \left( \frac{b}{a} \right) \right)^{(m+1)\nu+1}}{\Gamma((m+1)\nu+2)2^{(m+1)\nu}}.
\]


We can have many other interesting applications but we stop here.

**References**


