SOME RESULTS ON UNIVALENT HOLOMORPHIC FUNCTIONS BASED ON $q$-ANALOGUE OF NOOR OPERATOR

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Abstract: The main object of this paper is to define a new subclass of univalent holomorphic functions along with the recently defined $q$-analogue of Noor operator. We obtained a number of useful properties such as: coefficient bounds, extreme points, radii of starlikeness, convexity and close-to-convexity and weighted mean.

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1. Preliminaries

Let $A$ be the class of all functions $f(z)$ which are analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and have the following Taylor series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

Let us denote by $T$ the subclass of $A$ consisting of functions with negative coefficients of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \quad (2)$$

For functions $f$ and $g$ which are analytic in $U$ and have the form (2), we define ...
the convolution (or Hadamard product) of $f$ and $g$ by:

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U}).$$  \hspace{1cm} (3)

Now, we provide some notations regarding the $q$-calculus used in this article, see [1, 3] and [4].

For $0 < q < 1$, the $q$-derivative of $f$ is defined by:

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)} \quad (z \neq 0).$$  \hspace{1cm} (4)

We can easily conclude that:

$$D_q \left( \sum_{k=2}^{\infty} a_k z^k \right) = \sum_{k=2}^{\infty} [k, q] a_k z^{k-1} \quad (k \in \mathbb{N}, \ z \in \mathbb{U}),$$  \hspace{1cm} (5)

where

$$[k, q] = \frac{1 - q^k}{1 - q} = 1 + \sum_{t=1}^{k-1} q^t \quad ([0, q] = 0),$$  \hspace{1cm} (6)

and

$$[k, q] = \begin{cases} 1, & k = 0, \\ [1, q][2, q] \cdots [k, q], & k \in \mathbb{N}. \end{cases}$$  \hspace{1cm} (7)

Also, the $q$-generalization of the Pochhammer symbol for $y > 0$ is defined by:

$$[y, q]_k = \begin{cases} [y, q][y + 1, q] \cdots [y + k - 1, q], & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$  \hspace{1cm} (8)

For $\mu > -1$ and $f(z) \in \mathcal{T}$, we consider the $q$-analogue of Noor integral operator as follows:

$$\mathcal{N}_q^\mu f(z) = \mathcal{T}_{q, \mu+1}^{-1}(z) * f(z) = z - \sum_{k=2}^{\infty} \Psi_{k-1} a_k z^k \quad (z \in \mathbb{U}),$$  \hspace{1cm} (9)

where

$$\mathcal{T}_{q, \mu+1}^{-1}(z) * \mathcal{T}_{q, \mu+1}(z) = z D_q f(z),$$  \hspace{1cm} (10)
\[ T_{q,\mu+1}(z) = z - \sum_{k=2}^{\infty} \frac{[\mu + 1, q]_{k-1}}{[k, q]!} z^k, \quad (11) \]

and

\[ \Psi_{k-1} = \frac{[k, q]!}{[\mu + 1, q]_{k-1}}, \quad (12) \]

see [2].

It is clear that \( N_0^q f(z) = z D_q f(z), \) \( N_1^q f(z) = f(z) \)

\[ \lim_{q \to 1^-} N_{q}^{\mu} f(z) = z - \sum_{k=2}^{\infty} \frac{k!}{(\mu + 1)_{k-1}} a_k z^k, \quad (13) \]

which is the familiar Noor integral operator, see [5] and [6].

For \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta < 1, \) the function \( f(z) \in \mathcal{T} \) is in the class \( N_{q}^{\mu}(\alpha, \beta) \) if it satisfies:

\[ \text{Re} \left\{ \frac{z D_q (N_q^{\mu}(\alpha, \beta)) + \alpha z^2 D_q^2 (N_q^{\mu}(\alpha, \beta))}{\alpha z D_q (N_q^{\mu}(\alpha, \beta)) + (1 - \alpha) N_q^{\mu}(\alpha, \beta)} \right\} > \beta, \quad (14) \]

where \( D_q \) and \( N_q^{\mu} \) are defined in (4) and (9) respectively. Also \( D_q^2 (N_q^{\mu}(\alpha, \beta)) \) means \( D_q [D_q (N_q^{\mu}(\alpha, \beta))]. \)

**2. Main results**

In this section, we obtain coefficient bounds for functions in the class \( N_{q}^{\mu}(\alpha, \beta) \)

and show that this class is a convex set.

**Theorem 1.** \( f(z) \in \mathcal{T} \) is in the class \( N_{q}^{\mu}(\alpha, \beta) \) if and only if:

\[ \sum_{k=2}^{\infty} \Psi_{k-1} \left( [k, q](1 + \alpha[k, q] - \alpha \beta) + \beta(1 - \alpha) \right) a_k \leq 1 - \beta, \quad (15) \]

where \( \Psi_{k-1} \) and \( [k, q] \) are given by (12) and (6), respectively.

**Proof.** By making use of (4) and (5), we obtain:

\[ D_q (N_q^{\mu}(\alpha, \beta)) = 1 - \sum_{k=2}^{\infty} [k, q] \Psi_{k-1} a_k z^{k-1}, \quad (16) \]
\[ D_q^2 \left( \mathcal{N}_q^{\mu}(f(z)) \right) = -\sum_{k=2}^{\infty} [k, q]^2 \Psi_{k-1} a_k z^{k-2}, \quad (17) \]

where \([k, q]\) and \(\Psi_{k-1}\) are defined in (6) and (12), respectively.

By replacing (16) and (17) in (14) we have:

\[
\text{Re} \left\{ \frac{z - \sum_{k=2}^{\infty} [k, q] \Psi_{k-1} a_k z^k - \sum_{k=2}^{\infty} \alpha[k, q]^2 \Psi_{k-1} a_k z^k}{A} \right\} > \beta
\]

where

\[ A = \alpha z - \sum_{k=2}^{\infty} \alpha[k, q] \Psi_{k-1} a_k z^k + (1 - \alpha)z - \sum_{k=2}^{\infty} (1 - \alpha) \Psi_{k-1} a_k z^k, \]

or

\[
\text{Re} \left\{ \frac{z - \sum_{k=2}^{\infty} [k, q] \Psi_{k-1} (1 + \alpha[k, q]) a_k z_k}{z - \sum_{k=2}^{\infty} \Psi_{k-1} (\alpha([k, q] - 1) + 1) a_k z^k} \right\} > \beta.
\]

By choosing the values of \(z\) on the real axis and then letting \(z \to 1^-\) through real values, we get:

\[ 1 - \beta - \sum_{k=2}^{\infty} \left[ [k, q] \Psi_{k-1} (1 + \alpha[k, q]) - \beta \Psi_{k-1} (\alpha([k, q] - 1) + 1) \right] a_k \geq 0, \]

or

\[ \sum_{k=2}^{\infty} \Psi_{k-1} \left[ [k, q] (1 + \alpha[k, q] - \alpha\beta) - \beta(1 - \alpha) \right] a_k \leq 1 - \beta. \]

Conversely, suppose that (15) holds true. We will show that (14) is satisfies and so \(f \in \mathcal{N}_q^{\mu}(\alpha, \beta)\). Using the fact that \(\text{Re}\{W\} > \beta\) if and only if \(|W - (1 - \beta)| < |W - (1 - \beta)|\), it is enough to show that:

\[ L = \left| \frac{z D_q(\mathcal{N}_q^{\mu} f(z)) + \alpha z^2 D_q^2(\mathcal{N}_q^{\mu} f(z))}{\alpha z D_q(\mathcal{N}_q^{\mu} f(z)) + (1 - \alpha) \mathcal{N}_q^{\mu} f(z) - 1 - \beta} \right| \]
\[< \left| \frac{zD_q(\mathcal{N}_q^{\mu} f(z)) + \alpha z^2 D_q^2(\mathcal{N}_q^{\mu} f(z))}{\alpha zD_q(\mathcal{N}_q^{\mu} f(z)) + (1 - \alpha)\mathcal{N}_q^{\mu} f(z)} + 1 - \beta \right| = R. \]

But, if \(\alpha zD_q(\mathcal{N}_q^{\mu} f(z)) + (1 - \alpha)\mathcal{N}_q^{\mu} f(z) = J\), then we have:

\[L = \frac{1}{|J|} \left[ zD_q(\mathcal{N}_q^{\mu} f(z)) + \alpha z^2 D_q^2(\mathcal{N}_q^{\mu} f(z)) - (1 + \beta)J \right]. \]

By (16) and (17) we get:

\[L = \frac{1}{|J|} \left[ \beta z - \sum_{k=2}^{\infty} \Psi_{k-1} [k, q] (1 + \alpha[k, q] + (1 - \beta)) \right.\]

\[+ (1 - \beta)(1 - \alpha) a_k z^k \left. \right] < \frac{|z|}{|J|} \left[ \beta + \sum_{k=2}^{\infty} \Psi_{k-1} [k, q] (1 + \alpha[k, q] + (1 - \beta)) \right.\]

\[+ (1 - \beta)(1 - \alpha) a_k |z|^{k-1} \right]. \]

and

\[R = \frac{1}{|J|} \left| zD_q(\mathcal{N}_q^{\mu} f(z)) + \alpha z^2 D_q^2(\mathcal{N}_q^{\mu} f(z)) + (1 - \beta)J \right| \]

\[= \frac{1}{|J|} \left| (2 - \beta)z - \sum_{k=2}^{\infty} \Psi_{k-1} [k, q] (1 + \alpha[k, q] + (1 - \beta)) \right.\]

\[+ (1 - \beta)(1 - \alpha) a_k z^k \left. \right| \]

\[\geq \frac{|z|}{|J|} \left[ (2 - \beta) - \sum_{k=2}^{\infty} \Psi_{k-1} [k, q] (1 + \alpha[k, q] + (1 - \beta)) \right.\]

\[+ (1 - \beta)(1 - \alpha) a_k |z|^{k-1} \right]. \]

When \(z \in \partial U = \{z \in \mathbb{C} : |z| = 1\}\), it is easy to verify that \(R - L > 0\), if (15) holds and so the proof is complete. \(\square\)
Remark. The result (15) is sharp for the function \( F(z) \) given by:
\[
F(z) = z - \frac{1 - \beta}{\Psi_1([2, q][k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))} z^2,
\]
where \( \Psi_1 = \frac{[2, q]^!}{[\mu + 1, q]^!} \) and \([2, q] = 1 + q\).

Corollary 2. If \( f(z) \in \mathcal{N}_q^\mu(\alpha, \beta) \), then for \( k = 1, 2, \ldots \), we have:
\[
a_k \leq \frac{1 - \beta}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}.
\]

Theorem 3. \( \mathcal{N}_q^\mu(\alpha, \beta) \) is a convex set.

Proof. We must show that, if the functions \( f_t(z), t = 1, 2, \ldots, m, \) be in the class \( \mathcal{N}_q^\mu(\alpha, \beta) \), then the function \( h(z) = \sum_{t=1}^{m} \lambda_t f_t(z) \) for \( \lambda_t \) and \( \sum_{t=1}^{m} \lambda_t = 1 \), is also in \( \mathcal{N}_q^\mu(\alpha, \beta) \).

By definition of \( h(z) \), we conclude:
\[
h(z) = \sum_{t=1}^{m} \lambda_t \left( z - \sum_{k=2}^{\infty} a_{k,t} z^k \right) = z - \sum_{k=2}^{\infty} \left( \sum_{t=1}^{m} \lambda_t a_{k,t} \right) z^k.
\]

But from Theorem 1, we have:
\[
\sum_{k=2}^{\infty} \Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)) \left( \sum_{t=1}^{m} \lambda_t a_{k,t} \right)
\]
\[
= \sum_{t=1}^{m} \lambda_t \left\{ \sum_{k=2}^{\infty} \Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)) a_{k,t} \right\}
\]
\[
\leq \sum_{t=1}^{m} \lambda_t (1 - \beta) = 1 - \beta,
\]
which completes the proof. \( \square \)
3. Extreme points and some properties of $\mathcal{N}_q^\mu(\alpha, \beta)$

In the last section, we obtain extreme points of $\mathcal{N}_q^\mu(\alpha, \beta)$ and investigate some properties of the same class.

**Theorem 4.** Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{(1 - \beta)z^k}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta + \beta(1 - \alpha)))},$$

where $k = 2, 3, \ldots$. Then $f \in \mathcal{N}_q^\mu(\alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} t_k f_k(z)$, where $t_k \geq 0$ and $\sum_{k=1}^{\infty} t_k = 1$. In particular, the extreme points of $\mathcal{N}_q^\mu(\alpha, \beta)$ are functions $f_1(z)$ and $f_k(z)$, where $k = 2, 3, \ldots$.

**Proof.** Let $f$ be expressed as in the above. This means that we can write:

$$f(z) = \sum_{k=1}^{\infty} t_k f_k(z) = t_1 f_1(z) + \sum_{k=2}^{\infty} t_k f_k(z)$$

$$= t_1 z + \sum_{k=2}^{\infty} t_k z$$

$$- \sum_{k=2}^{\infty} \frac{(1 - \beta)t_k}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta + \beta(1 - \alpha)))} z^k$$

$$= z \sum_{k=1}^{\infty} t_k - \sum_{k=2}^{\infty} d_k z^k,$$

where

$$d = \frac{(1 - \beta)t_k}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta + \beta(1 - \alpha)))}.$$

Therefore $f \in \mathcal{N}_q^\mu(\alpha, \beta)$ since by Theorem 1, we have:

$$\sum_{k=2}^{\infty} \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta + \beta(1 - \alpha)))}{1 - \beta} d_k$$

$$= \sum_{k=2}^{\infty} t_k = 1 - t_1 < 1.$$

Conversely, suppose that $f \in \mathcal{N}_q^\mu(\alpha, \beta)$. Then by (19), for $k = 2, 3, \ldots$, we have:

$$a_k \leq \frac{1 - \beta}{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta + \beta(1 - \alpha)))}.$$
By putting
\[ t_k = \frac{\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha))}{1 - \beta}, \quad (k \geq 2), \]
we have \( t_k \geq 0 \) and if \( t_1 = 1 - \sum_{k=2}^{\infty} t_k \), we get the required result. So the proof is complete. \( \square \)

**Theorem 5.** Let the function \( f(z) \) by (2) be in the class \( N_\mu^q(\alpha, \beta) \), then:

1. \( f(z) \) is starlike of order \( \delta_1 \) for \( 0 \leq \delta_1 < 1 \) in \( |z| < R_1 \),

\[ R_1 = \inf_k \left[ \frac{B}{(k - \sigma_1)(1 - \beta)} \right]^{\frac{1}{k-1}}, \quad (20) \]

where
\[ B = (1 - \delta_1)\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)). \]

2. \( f(z) \) is convex of order \( \delta_2 \) for \( 0 \leq \delta_2 < 1 \) in \( |z| < R_2 \), where:

\[ R_2 = \inf_k \left[ \frac{C}{k(k - 2\delta_2)(1 - \beta)} \right]^{\frac{1}{k-1}}, \quad (21) \]

where
\[ C = (1 - \delta_2)\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)). \]

3. \( f(z) \) is close-to-convex of order \( \delta_3 \) for \( 0 \leq \delta_3 < 1 \) in \( |z| < R_3 \), where:

\[ R_3 = \inf_k \left[ \frac{D}{k(1 - \beta)} \right]^{\frac{1}{k-1}}, \quad (22) \]

where
\[ D = (1 - \delta_3)\Psi_{k-1}([k, q](1 + \alpha[k, q] - \alpha\beta) + \beta(1 - \alpha)). \]

**Proof.** To establish the required result, it is sufficient to prove that:

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta_1, \quad (|z| \leq R_1). \]
But

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| = \frac{\left| z - \sum_{k=2}^{\infty} ka_k z^k \right|}{\left| z - \sum_{k=2}^{\infty} a_k z^k \right|} - 1 \]

\[ = \left| \frac{- \sum_{k=2}^{\infty} (k - 1) a_k z^k}{z - \sum_{k=2}^{\infty} a_k z^k} \right| \leq \frac{\sum_{k=2}^{\infty} (k - 1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} \leq 1 - \delta_1. \]

Thus

\[ \sum_{k=2}^{\infty} \left( \frac{k-\delta_1}{1-\delta_0} \right) a_k |z|^{k-1} \leq 1. \]

Since \( f(z) \in N^\mu_q(\alpha, \beta) \), the last inequality holds, if:

\[ |z|^{k-1} \leq \left[ \frac{D}{(k - \delta_1)(1 - \beta)} \right], \]

where

\[ D = (1 - \delta_1) \Psi_{k-1} ([k, q] (1 + \alpha [k, q] - \alpha \beta) + \beta (1 - \alpha)). \]

In the last theorem, we investigate the weighted mean concept.

**Theorem 6.** If \( f \) and \( g \) belong to \( N^\mu_q(\alpha, \beta) \), then the weighted mean of \( f \) and \( g \) is also in the same class.

**Proof.** We have to prove that \( h_t(z) = \left[ \frac{\left( (1-t)f(z) + (1+t)g(z) \right)}{2} \right] \) is in the class \( N^\mu_q(\alpha, \beta) \).

Since \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=2}^{\infty} b_k z^k \), so:

\[ h_t(z) = z - \sum_{k=2}^{\infty} \left\{ \frac{(1 - t) a_k + (1 + t) b_k}{2} \right\} z^k. \]

To prove \( h_t(z) \in N^\mu_q(\alpha, \beta) \), by (15) we need to show that:

\[ \sum_{k=2}^{\infty} \frac{E}{2(1 - \beta)} < 1, \]

where

\[ E = \Psi_{k-1} ([k, q] (1 + \alpha [k, q] - \alpha \beta) + \beta (1 - \alpha)) \]
\times [(1 - t)a_k + (1 - t)b_k].

For this, we have:

\[ F = \sum_{k=2}^{\infty} \frac{E}{2(1 - \beta)} \Psi_{k-1}[k, q](1 + \alpha[k, q] - \alpha \beta + \beta(1 - \alpha)\frac{1}{1 - \beta}a_k
\]

\[ + \frac{1}{2} \sum_{k=2}^{\infty} \Psi_{k-1}[k, q](1 + \alpha[k, q] - \alpha \beta + \beta(1 - \alpha)\frac{1}{1 - \beta}b_k
\]

and by (15), we have:

\[ F < \frac{(1 - t)}{2} + \frac{(1 + t)}{2} = 1.
\]

Hence the result follows. \[\Box\]

References


