

## INITIAL-BOUNDARY VALUE PROBLEM FOR A SUBDIFFUSION EQUATION WITH CAPUTO DERIVATIVE

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**Abstract:** We investigate an initial-boundary value problem for a time-fractional subdiffusion equation with the Caputo derivatives on  $N$ -dimensional torus by the classical Fourier method. Since our solution is established on the eigenfunction expansion of elliptic operator, the method proposed in this article can be used to an arbitrary domain and an elliptic operator with variable coefficients. It should be noted that the conditions for the existence of a solution to the initial-boundary value problem found in the article cannot be weakened, and the article provides a corresponding example.

**AMS Subject Classification:** 35R11, 74S25

**Key Words:** time-fractional subdiffusion equation; Caputo derivatives; initial-boundary value problem; Fourier method; Liouville spaces

### 1. Main result

The fractional derivative in the sense of Caputo of order  $0 < \rho < 1$  of the function  $h(t)$  defined on  $[0, \infty)$  has the form (see, for example, [1], p. 14)

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{\frac{d}{d\tau} h(\tau) d\tau}{(t-\tau)^\rho}, \quad t > 0,$$

provided the right-hand side exists, and  $\Gamma(\sigma)$  is the Euler Gamma function. In

this definition, the differentiation and fractional integration are interchanged, compared with the original definition of the Riemann-Liouville derivative:

$$\partial_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dt} \int_0^t \frac{h(\tau) d\tau}{(t-\tau)^\rho}, \quad t > 0.$$

Note that if  $\rho = 1$ , then fractional derivatives coincide with the ordinary classical derivative of the first order:  $\partial_t h(t) = D_t h(t) = \frac{d}{dt} h(t)$ .

Let  $T^N$  be  $N$ -dimensional torus. Here,  $T^N = (-\pi, \pi]^N$ ,  $N \geq 1$ . We define by  $C(T^N)$  and  $C^2(T^N)$  a class of  $2\pi$ -periodic on each variable  $x_j$  functions  $v(x)$  from  $C(T^N)$  and  $C^2(T^N)$  correspondingly. Let  $A$  denote a positive operator, defined on  $C^2(T^N)$  and acting as  $Av(x) = -\Delta v(x)$ , where  $\Delta$  is the Laplace operator.

Let  $\rho \in (0, 1)$  be a constant number. Consider the initial-boundary value problem

$$D_t^\rho u(x, t) + Au(x, t) = f(x, t), \quad x \in T^N, \quad 0 < t \leq T, \quad (1)$$

$$u(x, 0) = \varphi(x), \quad x \in T^N, \quad (2)$$

where  $f$  and  $\varphi$  are given continuous functions.

**Definition 1.** A function  $u(x, t) \in C(T^N \times [0, T])$  with the properties  $D_t^\rho u(x, t), A(x, D)u(x, t) \in C(T^N \times (0, T])$ , and satisfying the conditions of problem (1) - (2) is called *the classical solution* of the initial-boundary value problem.

Initial-boundary value problems (1)-(2) for various elliptic operators  $A$  have been considered by a number of authors using different methods (see, for example, handbook [2]). In the book by A.A. Kilbas et al. [3] (Ch. 6), there is a survey on works published before 2006. The cases of one spatial variable  $x \in R$  and subdiffusion equation with the elliptical part  $u_{xx}$  were considered for example in the monograph of A.V. Pskhu [1] (Ch. 4, see references therein).

The main method used in this work is the Fourier method. As far as we know, in all previous papers this method was used to prove the existence of a generalized solution of initial-boundary value problems for subdiffusion equations. For example, in the paper of Yu. Luchko [4], the author constructed solutions by the eigenfunction expansion in the case of  $f = 0$  and discussed the unique existence of the generalized solution to problem (1)-(2) with the Caputo derivative. In an arbitrary  $N$ -dimensional domain  $\Omega$  initial-boundary value problems for subdiffusion equations (the fractional part of the equation is

a multi-term and initial conditions are non-local) with the Caputo derivatives has been investigated by M. Ruzhansky et al. [5]. The authors proved the existence and uniqueness of the generalized solution to the problem by the Fourier method. The authors of the paper [6] investigated both problem (1)-(2) and the corresponding backward problem with an arbitrary elliptic operator of the second order. To prove the existence and uniqueness of the generalized solution, the the Fourier method was applied.

The authors of papers [7], [8] and [9] used the Fourier method to construct a classical solution of the subdiffusion equations with the Riemann-Liouville derivative and various elliptic operators.

Let us introduce some concepts and formulate the main result of the work.

Let  $\hat{A}$  stand for the closure of operator  $A$  in  $L_2(T^N)$ . Then  $\hat{A}$  is self-adjoint and it has a complete (in  $L_2(T^N)$ ) set of eigenfunctions  $\{\gamma e^{inx}\}$ ,  $\gamma = \gamma(N) = (2\pi)^{-N/2}$ ,  $n \in Z^N$  and corresponding eigenvalues  $|n|^2 = n_1^2 + n_2^2 + \dots + n_N^2$ . Therefore, by virtue of J. von Niemann theorem, for any  $\tau > 0$  one can introduce the power of operator  $\hat{A}$  as  $\hat{A}^\tau g(x) = \sum_{n \in Z^N} |n|^\tau g_n e^{inx}$ , where  $g_n$  is Fourier coefficients:

$$g_n = (2\pi)^{-N} \int_{T^N} g(x) e^{-inx} dx.$$

The domain of definition of this operator is defined from the condition  $\hat{A}^\tau g(x) \in L_2(T^N)$  and has the form

$$D(\hat{A}^\tau) = \{g \in L_2(T^N) : \sum_{n \in Z^N} |n|^{2\tau} |g_n|^2 < \infty\}.$$

If we denote by  $L_2^a(T^N)$ ,  $a > 0$ , the Liouville space with the norm

$$\|g\|_{L_2^a(T^N)}^2 = \left\| \sum_{n \in Z^N} (1 + |n|^2)^{\frac{a}{2}} g_n e^{inx} \right\|_{L_2(T^N)}^2 = \sum_{n \in Z^N} (1 + |n|^2)^a |g_n|^2,$$

then it is not hard to see that  $D(\hat{A}^\tau) = L_2^m(T^N)$ .

Let  $E_{\rho, \mu}$  be the two-parametric Mittag-Leffler function:

$$E_{\rho, \mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

Here, is the main result.

**Theorem 2.** *Let  $a > \frac{N}{2}$  and  $\varphi \in L_2^a(T^N)$ . Moreover, let  $f(x, t) \in L_2^a(T^N)$  for  $t \in [0, T]$  and  $\|f(\cdot, t)\|_{L_2^a(T^N)}^2 \in C[0, T]$ . Then, there exists a solution of initial-boundary value problem (1)-(2) and it has the form*

$$u(x, t) = \sum_{n \in \mathbb{Z}^N} \left[ \varphi_n E_{\rho, 1}(-|n|^2 t^\rho) + \int_0^t f_n(t - \xi) \xi^{\rho-1} E_{\rho, \rho}(-|n|^2 \xi^\rho) d\xi \right] e^{inx}, \quad (3)$$

which converges absolutely and uniformly on  $x \in T^N$  and for each  $t \in (0, T]$ . Here,  $\varphi_n$  and  $f_n(t)$  are corresponding Fourier coefficients. Moreover, the series obtained after applying term-wise the operators  $D_t^\rho$  and  $A$  also converge absolutely and uniformly on  $x \in T^N$  and for each  $t \in (0, T]$ .

**Remark 3.** When  $a > \frac{N}{2}$ , according to the Sobolev embedding theorem, all functions in  $L_2^a(T^N)$  are  $2\pi$ -periodic continuous functions. The fulfillment of the inverse inequality  $a \leq \frac{N}{2}$ , admits the existence of unbounded functions in  $L_2^a(T^N)$  (see, for example, [10]). Therefore, condition  $a > \frac{N}{2}$  for function  $f$  of this theorem is not only sufficient for the statement to be hold, but it is also necessary.

A result similar to Theorem 2 was obtained in the recent paper [8] for a subdiffusion equation with the Riemann-Liouville derivative. But the condition found for the initial function  $\varphi(x)$  in that work is less restrictive. For example, in the one-dimensional case, it has been proved that it is sufficient to require  $\varphi(x) \in C(T^1)$ . As will be proved at the end of this paper, even Hlder continuous with exponent  $a = \frac{1}{2}$  for the initial function is not sufficient for the validity of Theorem 2 in the one-dimensional case.

In conclusion, note that since our solution is established on the eigenfunction expansion of elliptic operator, the method proposed in this article can be used to an arbitrary domain and an elliptic operator with variable coefficients.

## 2. Uniqueness

In this section, we prove the uniqueness of the solution to problem (1)-(2). Note that in order for problem (1)-(2) to have a unique solution, it is sufficient that the functions  $f$  and  $\varphi$  be continuous.

Suppose that initial-boundary value problem (1)-(2) has two classical solutions  $u_1(x, t)$  and  $u_2(x, t)$ . Our aim is to prove that  $u(x, t) = u_1(x, t) - u_2(x, t) \equiv$

0. Since the problem is linear, then we have the following homogenous problem for  $u(x, t)$ :

$$D_t^\rho u(x, t) + A u(x, t) = 0, \quad x \in T^N, \quad t > 0, \quad (4)$$

$$u(x, 0) = 0, \quad x \in T^N. \quad (5)$$

Let  $u(x, t)$  be a solution of problem (4)-(5). Consider the function

$$w_n(t) = \int_{T^N} u(x, t) e^{inx} dx, \quad n \in Z^N. \quad (6)$$

By virtue of equation (4), we can write

$$D_t^\rho w_n(t) = \int_{T^N} D_t^\rho u(x, t) e^{inx} dx = - \int_{T^N} A u(x, t) e^{inx} dx, \quad t > 0,$$

or, integrating by parts

$$D_t^\rho w_k(t) = - \int_{T^N} u(x, t) A e^{inx} dx = -|n|^2 \int_{T^N} u(x, t) e^{inx} dx = -|n|^2 w_n(t),$$

where  $t > 0$ .

Using in (6) the homogenous initial condition (5), we have the following Cauchy problem for  $w_n(t)$ :

$$D_t^\rho w_n(t) + |n|^2 w_n(t) = 0, \quad t > 0, \quad w_n(0) = 0.$$

This problem has the unique solution; therefore, the function defined by (6) is identically zero:  $w_n(t) \equiv 0$  (see for example, [1] p. 17, [11]). From the completeness in  $L_2(T^N)$  of the system of eigenfunctions  $\{e^{inx}\}$ , we have  $u(x, t) = 0$  for all  $x \in T^N$  and  $t > 0$ . Hence, the uniqueness is proved.

### 3. Existence

The proof of the existence is based on the following lemma (see M.A. Krasnosel'ski et al. [12], p. 453), which is a simple corollary of the Sobolev embedding theorem.

**Lemma 4.** *Let  $\sigma > 1 + \frac{N}{4}$ . Then for any  $|\alpha| \leq 2$  operator  $D^\alpha(\hat{A} + 1)^{-\sigma}$  (completely) continuously maps from  $L_2(T^N)$  into  $C(T^N)$  and moreover the following estimate holds true*

$$\|D^\alpha(\hat{A} + 1)^{-\sigma} g\|_{C(T^N)} \leq C \|g\|_{L_2(T^N)}.$$

A proof of this lemma can be found in [8].

One can easily verify that function (3) formally satisfies the conditions of problem (1)-(2). In order to prove that function (3) is actually a solution to the problem, it remains to substantiate this formal statement, i.e. show that the operators  $A$  and  $D_t^\rho$  can be applied term by term to series (3). To do this we remind the following asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument  $t$  and an arbitrary complex number  $\mu$  (see, for example, [13], p. 134)

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \quad (7)$$

We will also use a coarser estimate with a positive  $\lambda$  and  $0 < \varepsilon < 1$ :

$$|t^{\rho-1} E_{\rho,\mu}(-\lambda t^\rho)| \leq \frac{C t^{\rho-1}}{1 + \lambda t^\rho} \leq C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}, \quad t > 0, \quad (8)$$

which is easy to verify. Indeed, if  $t^\rho \lambda < 1$ , then  $t < \lambda^{-1/\rho}$  and

$$t^{\rho-1} = t^{\rho-\varepsilon\rho} t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$

If  $t^\rho \lambda \geq 1$ , then  $\lambda^{-1} \leq t^\rho$  and

$$\lambda^{-1} t^{-1} = \lambda^{-1+\varepsilon} \lambda^{-\varepsilon} t^{-1} \leq \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$

Note that the series (3) is in fact the sum of two series. Consider the following partial sums of the first series:

$$S_k^1(x, t) = \sum_{|n|^2 < k} E_{\rho,1}(-|n|^2 t^\rho) \varphi_n e^{inx}, \quad (9)$$

and suppose that function  $\varphi$  satisfies the condition of Theorem 2, i.e. for some  $\tau > \frac{N}{4}$

$$\sum_{n \in \mathbb{Z}^N} |n|^{4\tau} |\varphi_n|^2 \leq C_\varphi < \infty.$$

Since  $\hat{A}^{-\tau-1} e^{inx} = |n|^{-2(\tau+1)} e^{inx}$ , we may rewrite the sum (9) as

$$S_k^1(x, t) = \hat{A}^{-\tau-1} \sum_{|n|^2 < k} E_{\rho,1}(-|n|^2 t^\rho) \varphi_n |n|^{2(\tau+1)} e^{inx}.$$

Therefore, by virtue of Lemma 4 one has

$$\|D^\alpha S_k^1\|_{C(T^N)} = \|D^\alpha \hat{A}^{-\tau-1} \sum_{|n|^2 < k} E_{\rho,1}(-|n|^2 t^\rho) \varphi_n |n|^{2(\tau+1)} e^{inx}\|_{C(T^N)}$$

$$\leq C \left\| \sum_{|n|^2 < k} E_{\rho,1}(-|n|^2 t^\rho) \varphi_n |n|^{2(\tau+1)} e^{inx} \right\|_{L_2(T^N)}. \quad (10)$$

Using the orthonormality of the system  $\{e^{inx}\}$ , we have

$$\|D^\alpha S_k^1\|_{C(T^N)}^2 \leq C \sum_{|n|^2 < k} |E_{\rho,1}(-|n|^2 t^\rho) \varphi_n |n|^{2(\tau+1)}|^2.$$

Application of estimate (7) and inequality  $|n|^2(1 + |n|^2 t^\rho)^{-1} < t^{-\rho}$  gives

$$\sum_{|n|^2 < k} |E_{\rho,1}(-|n|^2 t^\rho) \varphi_n |n|^{2(\tau+1)}|^2 \leq C t^{-2\rho} \sum_{|n|^2 < k} |n|^{4\tau} |\varphi_n|^2 \leq C t^{-2\rho} C_\varphi.$$

Therefore, we can rewrite estimate (11) as

$$\|D^\alpha S_k^1\|_{C(T^N)}^2 \leq C t^{-2\rho} C_\varphi.$$

This implies uniformly on  $x \in T^N$  convergence of differentiated sum (9) with respect to the variables  $x_j$  for each  $t \in (0, T]$ . On the other hand, sum (10) converges for any permutation of its members as well, since these terms are mutually orthogonal. This implies the absolute convergence of the differentiated sum (9) on the same interval  $t \in (0, T]$ .

Now, we consider the second part of series (3):

$$S_k^2(x, t) = \sum_{|n|^2 < k} \int_0^t f_n(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-|n|^2 \xi^\rho) d\xi e^{inx} \quad (11)$$

and suppose that function  $f(x, t)$  satisfies all the conditions of Theorem 2, i.e. the following series converges uniformly on  $t \in [0, T]$  for some  $\tau > \frac{N}{4}$ :

$$\sum_{n \in Z^N} |n|^{4\tau} |f_n(t)|^2 \leq C_f < \infty.$$

We choose a small  $\varepsilon > 0$  in such a way, that  $\tau + 1 - \varepsilon > 1 + \frac{N}{4}$ . Since  $\hat{A}^{-\tau-1+\varepsilon} e^{inx} = |n|^{-2(\tau+1-\varepsilon)} e^{inx}$ , we may rewrite sum (11) as

$$S_k^2(x, t) = \hat{A}^{-\tau-1+\varepsilon} \sum_{|n|^2 < k} \int_0^t f_n(t - \xi) \xi^{\rho-1} E_{\rho,\rho}(-|n|^2 \xi^\rho) d\xi |n|^{2(\tau+1-\varepsilon)} e^{inx}.$$

Then by virtue of Lemma 4 one has

$$\|D^\alpha S_k^2\|_{C(T^N)}$$

$$\begin{aligned}
&= \|D^\alpha \hat{A}^{-\tau-1+\varepsilon} \sum_{|n|^2 < k} \int_0^t f_n(t-\xi) \xi^{\rho-1} E_{\rho,\rho}(-|n|^2 \xi^\rho) d\xi |n|^{2(\tau+1-\varepsilon)} e^{inx}\|_{C(T^N)} \\
&\leq C \left\| \sum_{|n|^2 < k} \int_0^t f_n(t-\xi) \xi^{\rho-1} E_{\rho,\rho}(-|n|^2 \xi^\rho) d\xi |n|^{2(\tau+1-\varepsilon)} e^{inx} \right\|_{L_2(T^N)}.
\end{aligned}$$

Using the orthonormality of the system  $\{e^{inx}\}$ , we will have

$$\|D^\alpha S_k^2\|_{C(T^N)}^2 \leq C \sum_{|n|^2 < k} \left| \int_0^t f_n(t-\xi) \xi^{\rho-1} E_{\rho,\rho}(-|n|^2 \xi^\rho) d\xi |n|^{2(\tau+1-\varepsilon)} \right|^2.$$

Now we use estimate (8) and apply the generalized Minkowski inequality. Then, the following double inequality holds:

$$\|D^\alpha S_k^2\|_{C(T^N)}^2 \leq C \left( \int_0^t \xi^{\varepsilon\rho-1} \left( \sum_{|n|^2 < k} |n|^{4\tau} |f_n(t-\xi)|^2 \right)^{1/2} d\xi \right)^2 \leq C \cdot C_f,$$

where  $C$  depends on  $T$  and  $\varepsilon$ . Hence, using the same argument as above, we see that differentiated sum (11) with respect to the variables  $x_j$  converges absolutely and uniformly on  $(x, t) \in T^N \times [0, T]$ .

Further, from equation (1) one has

$$D_t^\rho (S_k^1 + S_k^2) = -A(S_k^1 + S_k^2) + \sum_{|n|^2 < k} f_n(t) e^{inx}.$$

Absolutely and uniformly convergence of the latter series can be proved as above.

Thus, Theorem 2 is completely proved.

#### 4. Counterexample

In this section, we will discuss the importance of the condition  $a > \frac{N}{2}$  of Theorem 2. When  $a > \frac{N}{2}$  all functions in  $L_2^a(T^N)$  belong to  $C(T^N)$ . As noted in Remark 3, if  $a = \frac{N}{2}$ , then in the class  $L_2^a(T^N)$  there exist unbounded functions, as a consequence of which problem (1)-(2) certainly does not have classical solutions. Therefore, the question naturally arises: is it possible to replace, for example, condition  $\varphi \in L_2^a(T^N)$ ,  $a > \frac{N}{2}$ , of Theorem 2 by condition

$$\varphi \in L_2^{\frac{N}{2}}(T^N) \cap C(T^N)? \quad (12)$$

The following example answers this question in the negative.

Let  $N = 1$ . In the class of periodic functions, we seek a solution to the following problem

$$\begin{cases} D_t^\rho u(x, t) - u_{xx}(x, t) = 0, & 0 < t \leq T, \\ u(x, 0) = \varphi(x). \end{cases} \quad (13)$$

If  $\varphi$  satisfies the condition of Theorem 2, then the unique solution of the problem has the form

$$u(x, t) = \lim_{k \rightarrow \infty} \sum_{|n| \leq k} \varphi_n E_{\rho, 1}(-|n|^2 t^\rho) e^{inx}. \quad (14)$$

Recall that the classes  $C^a(T^1)$  are usually defined as follows:  $2\pi$ -periodic function  $\varphi \in C^a(T^1)$  if and only if

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^a.$$

Consider the following function, first studied by Hardy and Littlewood (see [14], proof of Theorem (3.10)):

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{e^{in \ln n}}{n} e^{inx}.$$

The real and imaginary parts of this function belong to the class  $C^{\frac{1}{2}}(T^1)$  (see [14], Chapter V, paragraph 4). Set  $\varphi(x) = \Re(\Phi(x))$ , where  $\Re(z)$  is a real part of the complex number  $z$ . Then  $\varphi \in C^{\frac{1}{2}}(T^1)$  and it is not hard to see, that

$$\sum_{n=1}^{\infty} \sqrt{(\varphi_n^c)^2 + (\varphi_n^s)^2} = +\infty,$$

where  $\varphi_n^c$  is the coefficients of the function  $\varphi(x)$  in  $\cos nx$  and  $\varphi_n^s$  - in terms of  $\sin nx$ . Obviously, the function  $\varphi(x)$  also belongs to the class (12) and if we denote  $\varphi_n = \frac{1}{2}(\varphi_n^c - i\varphi_n^s)$  and  $\varphi_{-n} = \overline{\varphi_n}$ , then

$$\sum_{n \in \mathbb{Z}^1} |\varphi_n| = +\infty. \quad (15)$$

Suppose that the solution to problem (13) has the form (14). Let us show that series (14) differentiated twice with respect to the variable  $x$  does not

converge absolutely, that is, for the function  $\varphi(x)$ , defined above, the statement of Theorem 2 does not hold. Indeed, set

$$(u_{k_0})_{xx}(x, t) = - \sum_{k_0 \leq |n|} \varphi_n E_{\rho,1}(-|n|^2 t^\rho) |n|^2 e^{inx}.$$

In order for this series to converge uniformly and absolutely with respect to  $x \in T^1$  and for each  $t \in (0, T]$  it is necessary that the number series

$$U_{k_0} = \sum_{k_0 \leq |n|} |\varphi_n E_{\rho,1}(-|n|^2 t^\rho)| |n|^2$$

converge for some  $t > t_0 > 0$ . Now we remind the following asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument

$$E_{\rho,1}(-|n|^2 t^\rho) = \frac{1}{\Gamma(1-\rho)} \cdot \frac{1}{|n|^2 t^\rho} + O\left(\frac{1}{|n|^2 t^\rho}\right)^2.$$

Hence, for sufficiently large  $k_0$ , we have

$$U_{k_0} = \frac{1}{t^\rho \Gamma(1-\rho)} \cdot \sum_{k_0 \leq |n|} |\varphi_n| + O(1),$$

where  $O(1)$  depends on  $t_0$  and  $\rho$ . Because of the equality (15) this series does not converge.

Thus, condition  $\varphi \in L_2^a(T^N)$ ,  $a > \frac{N}{2}$ , of Theorem 2 cannot be replaced by condition (12) in at least one-dimensional case.

In conclusion, note that a similar result with Theorem 2 for equation (1) with the Riemann-Liouville derivative is valid for all functions  $\varphi \in C(T^N) \cap L_2^{a-2}(T^N)$  (see [8]).

### Acknowledgement

The author convey thanks to R.R. Ashurov for discussions of these results. I also want to express my special thanks to the anonymous journal's reviewers for their comments, which have greatly improved the content of this article.

The author acknowledges financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

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