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A NOTE ON q-ANALOGUE OF POLY-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract: In this paper, we introduce a q-analogue of the poly-Genocchi numbers and polynomials which is a generalization of the poly-Genocchi numbers and polynomials including q-polylogarithm function. We also give the relation between generalized poly-Genocchi polynomials. We derive some relations that are connected with the Stirling numbers of the second kind. By using special functions, we investigate some symmetric identities involving q-poly-Genocchi polynomials.

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1. Introduction

Carlitz has introduced the q-Bernoulli numbers and polynomials in [1]. Srivastava and Pinter proved some relations and theorems between the Bernoulli

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polynomials and Euler polynomials in [18]. They also gave some generalizations of these polynomials. In [12]-[16], Kim et al. investigated some properties of the q-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [2], Cenkci et al. gave the q-extension of Genocchi numbers in a different manner. In [14], Kim gave a new concept for the q-Genocchi numbers and polynomials. In [17], Simsek et al. investigated the q-Genocchi zeta function and l-function by using generating functions and Mellin transformation.

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers,

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \longrightarrow 1} [x]_q = x$.

The classical Genocchi numbers G_n, the classical Genocchi polynomials $G_n(x)$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see [1]-[18]):

$$\left(\frac{2t}{e^t + 1}\right) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \qquad (|\ t\ | < \pi), \tag{1.1}$$

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)} \frac{t^n}{n!} \qquad (|\ t\ | < \pi), \tag{1.2}$$

and

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)} \frac{t^n}{n!} \qquad (|\ t\ | < \pi; 1^{\alpha} = 1), \tag{1.3}$$

with

$$G_n^1(0) \ = \ G_n(0) \ = \ G_n,$$

respectively.

Kim et al. [16] introduced and studied poly-Genocchi numbers and polynomials as follows:

$$\frac{2\mathrm{Li}_k(1 - e^{-t})}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}, n \in \mathbb{N}_0), \tag{1.4}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \quad |z| < 1, \quad \text{(see [4])}.$$

So for $k \leq 1$,

$$\mathrm{Li}_k(z) = -\ln(1-z), \quad \mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, ...$$

The k-th q-polylogarithm function $\mathrm{Li}_{k,q}$ is introduced by (see [3]):

$$\mathrm{Li}_{k,q}(z) = \sum_{m=1}^{\infty} \frac{z^m}{[m]_q^k} \quad (k \in \mathbb{Z}). \tag{1.5}$$

For nonnegative integer k, the q-polylogarithm function is represented by a rational function

$$\mathrm{Li}_{k,q}(z) = \frac{1}{(1-q)^k} \sum_{l=0}^{k} (-1)^l \left(\begin{array}{c} k \\ l \end{array} \right) \frac{q^l x}{1-q^l x}. \tag{1.6}$$

Recently, Jung-Ryoo [3] introduced q-analogue of poly-Bernoulli numbers and polynomials defined by means of the following generation function:

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}, n \in \mathbb{N}_0). \tag{1.7}$$

The Stirling number of the first kind is given by (see [15-19])

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l \quad (n \geq 0), \tag{1.8}$$

and the Stirling number of the second kind is defined by generating function

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}. \tag{1.9}$$

The main object of this paper is follows. We consider q-analogue of poly-Genocchi polynomials and we give some formulae of these polynomials related to the Stirling numbers of the second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means of the respective generating functions.

2. A q-analogue of poly-Genocchi polynomials

In this section, we define a q-analogue of poly-Genocchi polynomials $G_{n,q}^{(k)}(x)$ and investigate some properties of these polynomials. We start a following definition.

Definition 2.1. For $n \geq 0$, $n, k \in \mathbb{Z}$ and $0 \leq q < 1$, we introduce a q-analogue of poly-Genocchi polynomials by means of the following generating function:

$$\frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}, \tag{2.1}$$

where

$$\mathrm{Li}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^k}$$

is the k-th q-polylogarithm function.

When $x = 0$ in (2.1), $G_{n,q}^{(k)} = G_{n,q}^{(k)}(0)$ are called a q-analogue of poly-Genocchi numbers.

From (2.1), we have

$$\lim_{q\to 1} \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} \quad \text{(see [16])}. \tag{2.2}$$

For $k = 1$ and $q \to 1$ in (2.1), we have

$$\frac{2\mathrm{Li}_{1,1}(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{2.3}$$

Thus by (2.1) and (2.3), we get

$$G_{n,q}^{(k)}(x) = G_n(x) \quad (n \geq 0),$$

where $G_n(x)$ is the Genocchi polynomials (see [5]).

Theorem 2.1. *For $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,q}^{(k)} x^m. \tag{2.4}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q}^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^m\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix} G_{n-m,q}^{(k)}x^m\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we obtain (2.4). □

Theorem 2.2. *For $n \geq 0$, we have*

$$G_{n,1}^{(2)}(x) = \sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}\frac{B_m m!}{m+1}G_{n-m}(x). \tag{2.5}$$

Proof. Consider (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,1}^{(k)}(x)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{k,1}(1-e^{-t})}{e^t+1}e^{xt}$$

$$= \frac{2}{e^t+1}e^{xt}\int_0^t \frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t \frac{z}{e^z-1}dz\cdots dz.$$

In particular, if $k=2$, we have

$$\sum_{n=0}^{\infty} G_{n,1}^{(2)}(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}\int_0^t \frac{z}{e^z-1}dz = \left(\sum_{m=0}^{\infty}\frac{t^m B_m m!}{(m+1)m!}\right)\frac{2t}{e^t+1}e^{xt}$$

$$= \left(\sum_{m=0}^{\infty}\frac{t^m B_m m!}{(m+1)m!}\right)\left(\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}\frac{B_m m!}{m+1}G_{n-m}(x)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in the above equation, we get the result (2.5). □

Theorem 2.3. *For $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{[l]_q^k (p+1)} \left(\begin{array}{c} n \\ p \end{array} \right) G_{n-p}(x). \tag{2.6}$$

Proof. From equation (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x) \frac{t^n}{n!} = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t} \right) \left(\frac{2t}{e^t+1} e^{xt} \right).$$

Now

$$\frac{1}{t}\mathrm{Li}_{k,q}(1-e^{-t}) = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{[l]_q^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1-e^{-t})^l$$

$$= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{[l]_q{}^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p,l) \frac{t^p}{p!}$$

$$= \frac{1}{t} \sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{[l]_q^k} l! S_2(p,l) \frac{t^p}{p!}$$

$$= \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{[l]_q^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!},$$

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{[l]_q^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \right).$$

Replacing n by $n-p$ in the r.h.s of above equation and comparing the coefficients of t^n, we get the result (2.6). □

Theorem 2.4. *For $n \geq 1$, we have*

$$G_{n,q}^{(k)}(x+1) + G_{n,q}^{(k)}(x) = 2 \sum_{p=1}^{n} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{[l]_q^k} l! S_2(p,l) \left(\begin{array}{c} n \\ p \end{array} \right) x^{n-p}. \tag{2.7}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x+1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} G_{n.q}^{(k)}(x)\frac{t^n}{n!}$$

$$= \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{(x+1)t} + \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt}$$

$$= 2\mathrm{Li}_{k,q}(1-e^{-t})e^{xt}$$

$$= \sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{[l]_q^k}l!S_2(p,l)\right)\frac{t^p}{p!}e^{xt}$$

$$= \left(\sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{[l]_q^k}l!S_2(p,l)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of t^n, we get the result (2.7). □

Theorem 2.5. *For $d \in \mathbb{N}$ with $d \equiv 1(mod 2)$, we have*

$$G_{n,q}^{(k)}(x) = \sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{[l]_q^k} \times (-1)^a G_{n-p}(\frac{a+x}{d}). \tag{2.8}$$

Proof. From equation (2.1), we can be written as

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt} = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t}\right)\left(\frac{2t}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a e^{(a+x)t}\right)$$

$$= \left(\sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{[l]_q^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\right) \left(\sum_{m=0}^{\infty}d^{m-1}\sum_{a=0}^{d-1}(-1)^a G_n(\frac{a+x}{d})\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in above equation and comparing the coefficient of t^n, we get the result (2.8). □

3. Summation formulae for q-analogue of poly-Genocchi polynomials

In this section, we derive implicit formulae for q-analogue of poly-Genocchi polynomials $G_{n,q}^{(k)}(x)$ and some properties. We start a following theorem as follows.

Theorem 3.1. *For $x, y \in \mathbb{R}$ and $n \geq 0$, we have*

$$G_{l+p,q}^{(k)}(y) = \sum_{m,n=0}^{l,p} \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} (y-x)^{m+n} G_{l+p-m-n,q}^{(k)}(x). \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$\frac{2\mathrm{Li}_{k,q}(1-(e)^{-(t+u)})}{e^{t+u}+1} = e^{-x(t+u)} \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(x) \frac{t^l}{l!} \frac{u^p}{p!}. \tag{3.2}$$

Replacing x by y in the above equation and equating the resulting equation to the above equation, we get

$$e^{(y-x)(t+u)} \sum_{m,l=0}^{\infty} G_{l+p,q}^{(k)}(x) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(y,x) \frac{t^l}{l!} \frac{u^p}{p!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(y-x)(t+u)]^N}{N!} \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(x) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(y) \frac{t^l}{l!} \frac{u^p}{p!} \tag{3.4}$$

which on using the formula ([6])

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n}{n!} \frac{y^m}{m!} \tag{3.5}$$

in the left hand side, becomes

$$\sum_{m,n=0}^{\infty} \frac{(y-x)^{m+n} t^m u^n}{m!n!} \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(x) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(y) \frac{t^l}{l!} \frac{u^p}{p!}. \tag{3.6}$$

Now replacing l by $l-m$, p by $p-n$ in the left hand side of (3.6), we get

$$\sum_{m,n=0}^{\infty} \sum_{l,p=0}^{\infty} \frac{(y-x)^{m+n}}{m!n!} G_{l+p-m-n,q}^{(k)}(x) \frac{t^l}{(l-m)!} \frac{u^p}{(p-n)!}$$

$$= \sum_{l,p=0}^{\infty} G_{l+p,q}^{(k)}(y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

Corollary 3.1. *By taking $l = 0$ in equation (3.1), we have*

$$G_{p,q}^{(k)}(y) = \sum_{n=0}^{p} \binom{p}{n} (y-x)^{m+n} G_{p-n,q}^{(k)}(x). \tag{3.8}$$

Theorem 3.2. *For $x, y \in \mathbb{R}$ and $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x+u) = \sum_{j=0}^{n} \binom{n}{j} u^j G_{n-j,q}^{(k)}(x). \tag{3.9}$$

Proof. Since

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x+u)\frac{t^n}{n!} = \frac{\mathrm{Li}_{k,q}(1-(e)^{-t})}{e^t+1}e^{(x+u)t}$$

$$= \left(\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} u^j\frac{t^j}{j!}\right).$$

Now replacing n by $n-j$ and comparing the coefficients of t^n, we get the result (3.9). □

Theorem 3.3. *For $x \in \mathbb{R}$ and $n \geq 0$, we have*

$$G_{n,q}^{(k)}(x+1) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,q}^{(k)}(x). \tag{3.10}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt}(e^t-1)$$

$$= \left(\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right) - \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} G_{n-m,q}^{(k)}(x)\frac{t^n}{(n-m)!m!} - \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.10). □

Theorem 3.4. *For $x \in \mathbb{R}$ and $n \geq 0$, we have*

$$G_{n,q}^{(k)}(-x) = (-1)^n G_{n,q}^{(k)}(x). \tag{3.11}$$

Proof. We replace t by $-t$ in (2.1) and then subtract the result from (2.1) itself finding

$$\left[\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}(e^{xt}-e^{-xt})\right] = \sum_{n=0}^{\infty}[1-(-1)^n]G_{n,q}^{(k)}(x)\frac{t^n}{n!},$$

which is equivalent to

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} G_{n,q}^{(k)}(-x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]G_{n,q}^{(k)}(x)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!} - G_{n-m,q}^{(k)}(-x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]G_{n,q}^{(k)}(x)\frac{t^n}{n!},$$

and thus by equating coefficients of like powers of t^n, we get (3.11). □

4. Symmetry identities

In this section, we give general symmetry identities for q-analogue of poly-Genocchi polynomials $G_{n,q}^{(k)}(x)$ by applying the generating function (2.1). The results extend some known identities of Jung-Ryoo [3] and Khan et al. [5]-[9].

Theorem 4.1. *Let $a, b > 0$ and $a \neq b$ and $x, y \in \mathbb{R}$, $n \geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m} G_{n-m,q}^{(k)}(bx) G_{m,q}^{(k)}(ax)$$

$$= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^m b^{n-m} G^{(k)}_{n-m,q}(ax) G^{(k)}_{m,q}(bx). \tag{4.1}$$

Proof. Start with

$$G(t) = \left(\frac{(2\mathrm{Li}_{k,q}(1-e^{-at}))(2\mathrm{Li}_{k,q}(1-e^{-bt}))}{(e^{at}+1)(e^{bt}+1)} \right) e^{2abxt}. \tag{4.2}$$

Then $G(t)$ is symmetric in a and b and we can expand $G(t)$ into series in two ways to obtain

$$G(t) = \frac{1}{ab} \sum_{n=0}^{\infty} G^{(k)}_{n,q}(bx) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} G^{(k)}_{m,q}(ax) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^{n-m} b^m G^{(k)}_{n-m,q}(bx) G^{(k)}_{m,q}(ax) \right) \frac{t^n}{n!}.$$

Similarly, we can write

$$G(t) = \sum_{n=0}^{\infty} G^{(k)}_{n,q}(ax) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} G^{(k)}_{m,q}(bx) \frac{(at)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^m b^{n-m} G^{(k)}_{n-m,q}(ax) G^{(k)}_{m,q}(bx) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive the desired result. □

Corollary 4.1. *By setting $b = 1$ in Theorem 4.1, we have*

$$\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^{n-m} G^{(k)}_{n-m,q}(x) G^{(k)}_{m,q}(ax)$$

$$= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^m G^{(k)}_{n-m,q}(ax) G^{(k)}_{m,q}(x). \tag{4.3}$$

Theorem 4.2. *Let $a, b > 0$ and $a \neq b$, $n \geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} G^{(k)}_{n-m,q} \left(bx + \frac{b}{a} i + j \right) G^{(k)}_{m,q}(ay)$$

$$= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^m b^{n-m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} G_{n-m,q}^{(k)} \left(ax + \frac{a}{b} i + j \right) G_{m,q}^{(k)}(by). \tag{4.4}$$

Proof. Let

$$H(t) = \left(\frac{(2\mathrm{Li}_{k,q}(1-e^{-at}))(2\mathrm{Li}_{k,q}(1-e^{-bt}))(e^{abt}+1)^2 e^{ab(x+y)t}}{(e^{at}+1)^2(e^{bt}+1)^2} \right)$$

$$H(t) = \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-at})}{(e^{at}+1} \right) e^{abxt} \left(\frac{e^{abt}+1}{e^{bt}+1} \right)$$

$$\times \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-bt})}{e^{bt}+1} \right) e^{abyt} \left(\frac{e^{abt}+1}{e^{at}+1} \right)$$

$$= \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-at})}{(e^{at}+1} \right) e^{abxt} \sum_{i=0}^{a-1} (-1)^i e^{bti}$$

$$\times \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-bt})}{e^{bt}+1} \right) e^{abyt} \sum_{j=0}^{b-1} (-1)^j e^{atj} \tag{4.5}$$

$$= \left(\frac{2\mathrm{Li}_{k,q}(1-e^{-at})}{(e^{at}+1} \right) \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} e^{(bx+\frac{b}{a}i+j)at} \sum_{m=0}^{\infty} G_{m,q}^{(k)}(ay) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} G_{n,q}^{(k)} \left(bx + \frac{b}{a} i + j \right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} G_{m,q}^{(k)}(ay) \frac{(bt)^m}{(m)!},$$

$$H(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) b^m a^{n-m} \right.$$

$$\left. \times \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} G_{n-m,q}^{(k)} \left(bx + \frac{b}{a} i + j \right) G_{m,q}^{(k)}(ay) \right) \frac{t^n}{n!}. \tag{4.6}$$

On the other hand

$$H(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) a^m b^{n-m} \right.$$

$$\left. \times \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} G_{n-m,q}^{(k)} \left(ax + \frac{a}{b} i + j \right) G_{m,q}^{(k)}(by) \right) \frac{t^n}{n!}. \tag{4.7}$$

On comparing the coefficients of t^n on the right hand sides of the last two equations, we get (4.4). □

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