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STABILITY OF THE TIME-DEPENDENT IDENTIFICATION PROBLEM FOR THE DELAY HYPERBOLIC EQUATION

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Abstract: In the present paper, a time-dependent source identification problem for a one dimensional delay hyperbolic equation with Dirichlet condition is studied. Operator-functions generated by the positive operator are considered. Theorems on the stability estimates for the solution of this problem are established. The first order of accuracy difference scheme for this source identification problem is presented. Numerical analysis and discussions are presented.

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1. Introduction

There is permanently a major interest for the theory of source identification problems for partial differential equations since they have widespread applications in modern physics and technology. For this effort, the stability of various source identification problems for partial differential and difference equations has also been studied extensively by many researchers (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25] and the references given therein).

In many fields of the contemporary science and technology, systems with delaying terms appear. The dynamical processes are described by systems of delay ordinary and partial differential and difference equations. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed.

The stability of the delay differential and difference equations has been studied in many papers (see, e.g., [26], [27], [28], [29], [30], [31], [32], [33], [34], [35] and the references given therein).

In the present paper, the time-dependent identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \big(a(x)u_x(t,x)\big)_x = b\big(a(x)u_x(t-\omega)\big)_x \\ +p(t)q(x) + f(t,x), \\ 0 < t < \infty, x \in (0,l)\,, \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in [0,l]\,, \\ u(t,0) = u(t,l) = 0, \int\limits_0^l u(t,x)dx = \zeta\,(t)\,, t \ge 0 \end{cases} \tag{1}$$

for a one dimensional delay hyperbolic equation with Dirichlet condition is studied. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (1) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x), (t,x) \in (0,\infty) \times (0,l), g(t,x), (t,x) \in [-\omega,0] \times (0,l), a(x), q(x), x \in (0,l), \zeta\,(t)\,, t \in [0,\infty)$. Here b is a constant. Assume that $\int_0^l q(x)dx \neq 0$, and $q(0) = q(l) = 0$, and $g(t,0) = g(t,l) = 0, t \in [-\omega,0]\,, f(t,0) = f(t,l) = 0, t \in [0,\infty)\,, a(x) \ge a > 0, x \in (0,l)$.

This paper is organized as follows. Section 1 is introduction. In Section 2, operator-functions generated by the positive operator are considered. In Section 3, the main theorems on stability of differential problem (1) are established.

In Section 4, a first order of accuracy difference scheme for the numerical solution of source identification problem (1) is presented and numerical results are obtained. Finally, Section 5 contains conclusions.

2. Auxiliary statements

Let $c(t)$ is operator-function generated by the operator A and defined as the solution of the initial value problem for a second order differential equation

$$u_{tt}(t) + Au(t) = 0, 0 < t < \infty, u(0) = \varphi, u_t(0) = 0 \tag{2}$$

in a Hilbert space H, that is,

$$u(t) = c(t)\varphi.$$

Similarly, $s(t)$ is operator-function generated by the operator A and defined as the solution of the initial value problem for a second order differential equation

$$v_{tt}(t) + Av(t) = 0, \quad 0 < t < \infty, \ v(0) = 0, \ v_t(0) = \psi \tag{3}$$

in a Hilbert space H, namely,

$$v(t) = s(t)\psi.$$

By the definitions of $c(t)$ and $s(t)$, we have that

$$s'(t) = c(t), c'(t) = -As(t). \tag{4}$$

We consider the second order differential operator A determined by

$$Av = -\left(a\left(x\right)v_x(x)\right)_x \tag{5}$$

in $\mathbb{L}_2\left[0,l\right]$ with domain $\mathbb{D}\left(A\right) = \left\{v : v, v'' \in \mathbb{L}_2\left[0,l\right], v\left(0\right) = v\left(l\right) = 0\right\}$ dense in $\mathbb{L}_2\left[0,l\right]$. It is well-known that A is the positive-definite and self-adjoint operator in $\mathbb{L}_2\left[0,l\right]$. Let us give estimates (formula (6)) that will be needed below:

$$\begin{cases} \|A^{-\frac{1}{2}}\|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le l^{-\frac{1}{2}}, \ \|s(t)\|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le t, \\ \|c(t)\|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le 1, \quad \|A^{\frac{1}{2}}s(t)\|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le 1. \end{cases} \tag{6}$$

3. Stability of the differential equation problem (1)

Theorem 1. *Assume that* $\int\limits_0^l q(x)dx \neq 0$ *. Then for the solution of problem (1) the following stability estimate holds:*

$$\max_{0\leq t\leq\omega}|p(t)|,\ \max_{0\leq t\leq\omega}\|u_{tt}\|_{\mathbb{L}_2[0,l]},\ \max_{0\leq t\leq\omega}\|u_t\|_{\mathbb{W}_2^1[0,l]},\ \max_{0\leq t\leq\omega}\|u\|_{\mathbb{W}_2^2[0,l]} \tag{7}$$

$$\leq M(q,\alpha)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_{\mathbb{L}_2[0,l]}+\|f(0)\|_{\mathbb{L}_2[0,l]}+\max_{0\leq t\leq\omega}\left|\zeta''\right|\right],$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_{\mathbb{L}_2[0,l]},\ \max_{-\omega\leq t\leq 0}\|g_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$

$$\left.\max_{-\omega\leq t\leq 0}\|g(t)\|_{\mathbb{W}_2^2[0,l]}\right\},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}|p(t)|,\ \max_{n\omega\leq t\leq(n+1)\omega}\|u_{tt}\|_{\mathbb{L}_2[0,l]},\ \max_{n\omega\leq t\leq(n+1)\omega}\|u_t\|_{\mathbb{W}_2^1[0,l]}, \tag{8}$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|u\|_{\mathbb{W}_2^2[0,l]}\leq M(q,\alpha)\left[a_n+\max_{(n-1)\omega\leq t\leq n\omega}|p(t)|\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\left\|f'(t)\right\|_{\mathbb{L}_2[0,l]}+\|f(n\omega)\|_{\mathbb{L}_2[0,l]}+\max_{n\omega\leq t\leq(n+1)\omega}\left|\zeta''\right|\right],$$

$$a_n=\max\left\{\max_{(n-1)\omega\leq t\leq n\omega}\|u_{tt}(t)\|_{\mathbb{L}_2[0,l]},\ \max_{(n-1)\omega\leq t\leq n\omega}\|u_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega}\|u(t)\|_{\mathbb{W}_2^2[0,l]}\right\},\ n=1,2,\cdots.$$

Here $\mathbb{L}_2[0,l]$ *is the space of all square integrable functions* $w(x)$ *defined on* $[0,l]$ *and* $\mathbb{W}_2^k[0,l], k=1,2$ *are Sobolev spaces equipped with norms*

$$\|w\|_{\mathbb{W}_2^1[0,l]}=\left(\int_0^l\left[w^2(z)+w_z^2(z)\right]dz\right)^{\frac{1}{2}},$$

$$\|w\|_{\mathbb{W}_2^2[0,l]}=\left(\int_0^l\left[w^2(z)+w_{zz}^2(z)\right]dz\right)^{\frac{1}{2}},$$

respectively.

Proof. We will seek $u(t,x)$, using the substitution

$$u(t,\ x) = w(t,\ x) + \eta(t)q(x), \tag{9}$$

where $\eta(t)$ is the function defined by the formula

$$\begin{cases} \eta(t) = \int_{(n-1)\omega}^{t} (t-s)p(s)ds, \\ \eta((n-1)\,\omega) = \eta'((n-1)\,\omega) = 0, \quad n = 1, 2, \cdots . \end{cases} \tag{10}$$

It is easy to see that $w(t,\ x)$ is the solution of the problems

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2} - \frac{\partial^2 w(t,x)}{\partial x^2} = \eta(t)q''(x) + bg_{xx}(t-\omega, x) + f(t,x), \\ 0 < t < \omega,\ x \in (0,l)\,, \\ w(0,\ x) = g(0,x),\ w_t(0,\ x) = g_t(0,x),\ x \in (0,l)\,, \\ w(t,0) = w(t,l) = 0,\ t \geq 0 \end{cases} \tag{11}$$

and

$$\begin{cases} \frac{\partial^2 w(t,x)}{\partial t^2} - \frac{\partial^2 w(t,x)}{\partial x^2} = b\frac{\partial^2 w(t-\omega,x)}{\partial x^2} + (\eta(t) + b\eta(t-\omega))\, q''(x) \\ +f(t,x), (n-1)\,\omega < t < n\omega, \quad x \in (0,l)\,,\ n = 2, 3, \cdots , \\ w((n-1)\,\omega+,\ x) = w((n-1)\,\omega-,\ x), \\ w_t((n-1)\,\omega+,\ x) = w_t((n-1)\,\omega-,\ x), \\ x \in (0,l)\,,\ n = 2, 3, \cdots , \\ w(t,0) = w(t,l) = 0,\ t \geq 0. \end{cases} \tag{12}$$

Now we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition $\int\limits_0^l u(t,x)dx = \zeta(t)$ and substitution (9), we get

$$\eta(t) = \frac{\zeta(t) - \int\limits_0^l w(t,x)dx}{\int\limits_0^l q(x)dx}.$$

From that and $p(t)=\eta''(t)$, it follows that

$$p(t)=\frac{\zeta''(t)-\int\limits_0^l \frac{\partial^2}{\partial t^2}w(t,x)dx}{\int\limits_0^l q(x)dx}.$$

Then, using the triangle inequality, we obtain

$$|p(t)|\leq\frac{|\zeta''(t)|+\int\limits_0^l \left|\frac{\partial^2}{\partial t^2}w(t,x)\right|dx}{\left|\int\limits_0^l q(x)dx\right|} \tag{13}$$

$$\leq k(q,l)\left[\left|\zeta''(t)\right|+\left\|\frac{\partial^2}{\partial t^2}w(t,.)\right\|_{\mathbb{L}_2[0,l]}\right]$$

for all $t\in(0,\infty)$. Now, using substitution (9), we get

$$\frac{\partial^2 u\left(t,x\right)}{\partial t^2}=\frac{\partial^2 w\left(t,x\right)}{\partial t^2}+p(t)q(x).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u\left(t,\cdot\right)}{\partial t^2}\right\|_{\mathbb{L}_2[0,l]}\leq\left\|\frac{\partial^2 w\left(t,\cdot\right)}{\partial t^2}\right\|_{\mathbb{L}_2[0,l]}+|p(t)|\,\|q\|_{\mathbb{L}_2[0,l]} \tag{14}$$

for all $t\in(0,\infty)$. Therefore, the proof of Theorem 1 is based on the following theorem. □

Theorem 2. *Under the assumptions of Theorem 1, for the solution of problems (11) and (12) the following stability estimate holds:*

$$\max_{0\leq t\leq\omega}\|w_{tt}\|_{\mathbb{L}_2[0,l]},\ \max_{0\leq t\leq\omega}\|w_t\|_{\mathbb{W}_2^1[0,l]},\ \max_{0\leq t\leq\omega}\|w\|_{\mathbb{W}_2^2[0,l]} \tag{15}$$

$$\leq M\left(q,l\right)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_{\mathbb{L}_2[0,l]}+\left\|f\left(0\right)\right\|_{\mathbb{L}_2[0,l]}+\max_{0\leq t\leq\omega}\left|\zeta''\right|\right],$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_{\mathbb{L}_2[0,l]},\ \max_{-\omega\leq t\leq 0}\|g_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$

$$\max_{-\omega\leq t\leq 0}\left\|g(t)\right\|_{\mathbb{W}_2^2[0,l]}\Bigg\},$$

$$\max_{n\omega\leq t\leq (n+1)\omega}\left\|w_{tt}\right\|_{\mathbb{L}_2[0,l]},\ \max_{n\omega\leq t\leq (n+1)\omega}\left\|w_t\right\|_{\mathbb{W}_2^1[0,l]},\tag{16}$$

$$\max_{n\omega\leq t\leq (n+1)\omega}\left\|w\right\|_{\mathbb{W}_2^2[0,l]}\leq M\left(q,l\right)\Big[a_n$$

$$+\max_{n\omega\leq t\leq (n+1)\omega}\left\|f'(t)\right\|_{\mathbb{L}_2[0,l]}+\left\|f\left(n\omega\right)\right\|_{\mathbb{L}_2[0,l]}+\max_{n\omega\leq t\leq (n+1)\omega}\left|\zeta''\right|\Big],$$

$$a_n=\max\Bigg\{\max_{(n-1)\omega\leq t\leq n\omega}\left\|w_{tt}(t)\right\|_{\mathbb{L}_2[0,l]},\max_{(n-1)\omega\leq t\leq n\omega}\left\|w_t(t)\right\|_{\mathbb{W}_2^1[0,l]},$$

$$\max_{(n-1)\omega\leq t\leq n\omega}\left\|w(t)\right\|_{\mathbb{W}_2^2[0,l]}\Bigg\},\quad n=1,2,\cdots.$$

Proof. It is clear that the mixed problems (11) and (12) can be written as the initial value problems

$$\begin{cases} w''(t)+Aw\left(t\right)+\mu\left(t\right)Aq=bAg(t-\omega)\\ +f\left(t\right),\ t\in\left(0,\omega\right),\\ w\left(0\right)=g(0),\ w'\left(0\right)=g_t(0)\end{cases}\tag{17}$$

and

$$\begin{cases} w''(t)+Aw\left(t\right)+\mu\left(t\right)Aq=bAw(t-\omega)+f\left(t\right),\\ \left(n-1\right)\omega<t<n\omega,\ n=2,3,\cdots,\\ w\left(\left(n-1\right)\omega+\right)=w\left(\left(n-1\right)\omega-\right),\\ w'\left(\left(n-1\right)\omega+\right)=w'\left(\left(n-1\right)\omega-\right),\ n=2,3,\cdots\end{cases}\tag{18}$$

in a Hilbert space $\mathbb{H}=\mathbb{L}_2\left[0,l\right]$ with A determining by (5). From (10) and (13) it follows that

$$\left|p(t)\right|,\left|\mu\left(t\right)\right|\leq k(q,l)\left[\left|\zeta''(t)\right|+\left\|w_{tt}(t)\right\|_H\right]\tag{19}$$

for all $t\in\left(0,\infty\right)$. Therefore, the proof of Theorem 2 is based on the following abstract theorem. □

Theorem 3. *Under the assumptions of Theorem 1, for the solution of problems (17) and (18) the following stability estimate holds:*

$$\max_{0\leq t\leq\omega}\|w_{tt}\|_H,\ \max_{0\leq t\leq\omega}\left\|A^{\frac{1}{2}}w_t\right\|_H,\ \max_{0\leq t\leq\omega}\|Aw\|_H \tag{20}$$

$$\leq M(q,l)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_H+\|f(0)\|_H+\max_{0\leq t\leq\omega}\left|\zeta''\right|\right],$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_H,\ \max_{-\omega\leq t\leq 0}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H,\right.$$

$$\left.\max_{-\omega\leq t\leq 0}\|Ag(t)\|_H\right\},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|w_{tt}\|_H,\ \max_{n\omega\leq t\leq(n+1)\omega}\|w_t\|_H, \tag{21}$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|Aw\|_H\leq M(q,l)\left[a_n\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\left\|f'(t)\right\|_H+\|f(n\omega)\|_H+\max_{n\omega\leq t\leq(n+1)\omega}\left|\zeta''\right|\right],$$

$$a_n=\max\left\{\max_{(n-1)\omega\leq t\leq n\omega}\|w_{tt}(t)\|_H,\ \max_{(n-1)\omega\leq t\leq n\omega}\left\|A^{\frac{1}{2}}w_t(t)\right\|_H,\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega}\|Aw(t)\|_H\right\},\ n=1,2,\cdots.$$

Proof. The initial value problems (17) and (18) are equivalent to the integral equations

$$w(t)=c(t)\,g(0)+s(t)\,g_t(0) \tag{22}$$

$$+\int_0^t s(t-z)\left[-\mu(z)\,Aq+bAg(z-\omega)+f(z)\right]dz,\ 0\leq t\leq\omega$$

$$w(t)=c(t-(n-1)\,\omega)\,w((n-1)\,\omega) \tag{23}$$

$$+s(t-(n-1)\,\omega)\,w_t((n-1)\,\omega)$$

$$+\int_{(n-1)\omega}^t s(t-z)\left[-\mu(z)\,Aq+bAg(z-\omega)+f(z)\right]dz,$$

$$(n-1)\,\omega\leq t\leq n\omega,\ n=2,\cdots$$

in H, respectively. Let $t\in[0,\omega]$. Applying equation (17) and formula (22), we get

$$Aw(t)=c(t)\,Ag(0)+s(t)\,Ag_t(0)$$

$$+\int_0^t As\left(t-z\right)\left[-\mu\left(z\right)Aq+bAg(z-\omega)+f\left(z\right)\right]dz$$

$$=c\left(t\right)Ag(0)+s\left(t\right)Ag_t(0)$$

$$-\mu\left(t\right)Aq+bAg(t-\omega)+f\left(t\right)-c\left(t\right)\left[bAg(-\omega)+f\left(0\right)\right]$$

$$-\int_0^t c\left(t-z\right)\left[-\mu'\left(z\right)Aq+bAg'(z-\omega)+f'\left(z\right)\right]dz.$$

Therefore, applying this formula, the triangle inequality and estimates (6) and (19), we get

$$\|w_{tt}(t)\|_{\mathbb{H}}\leq\|Ag(0)\|_{\mathbb{H}}+\left\|A^{\frac{1}{2}}g_t(0)\ \right\|_{\mathbb{H}}+\|f\left(0\right)\|_{\mathbb{H}}+\omega\max_{t\in[0,\omega]}\|f_t\|_{\mathbb{H}}$$

$$+\max_{-\omega\leq t\leq 0}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H+M_3(q,l)\ \max_{0\leq t\leq\omega}\left|\zeta''\right|+M_3(q,l)\int_0^t\|w_{zz}(z)\|_H\,dz.$$

Using the integral inequality, we get

$$\max_{0\leq t\leq\omega}\|w_{tt}\|_H$$

$$\leq\ M\left(q,l\right)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_H+\|f\left(0\right)\|_H+\ \max_{0\leq t\leq\omega}\left|\zeta''\right|\right].$$

In the same manner, we can obtain

$$\max_{0\leq t\leq\omega}\left\|A^{\frac{1}{2}}w_t\right\|_H$$

$$\leq\ M\left(q,l\right)\left[a_0+\max_{0\leq t\leq\omega}\left\|f'(t)\right\|_H+\|f\left(0\right)\|_H+\ \max_{0\leq t\leq\omega}\left|\zeta''\right|\right].$$

From that and equation (17) it follows estimate for $\max\limits_{0\leq t\leq\omega}\|Aw\|_H$.

Let $t\in\left[(n-1)\,\omega,n\omega\right],n=2,\cdots$. Applying equation (18) and formula (23), we get

$$Aw(t)=c\left(t-(n-1)\,\omega\right)Aw((n-1)\,\omega)$$

$$+s\left(t-(n-1)\,\omega\right)Aw_t((n-1)\,\omega)$$

$$+\int_{(n-1)\omega}^t As\left(t-z\right)\left[-\mu\left(z\right)Aq+bAg(z-\omega)+f\left(z\right)\right]dz$$

$$=c\left(t\right)Aw((n-1)\,\omega)+s\left(t\right)Aw_t((n-1)\,\omega)$$

$$-\mu\left(t-(n-1)\,\omega\right)Aq+bAw(t-n\omega)$$

$$+f(t) - c(t - (n-1)\omega)\left[bAg(-n\omega) + f((n-1)\omega)\right]$$

$$-\int_{(n-1)\omega}^{t} c(t-z)\left[-\mu'(z)Aq + bAg'(z-\omega) + f'(z)\right]dz.$$

Therefore, applying this formula, the triangle inequality and estimates (6) and (19), we get

$$\|w_{tt}(t)\|_{\mathbb{H}} \leq \|Aw((n-1)\omega)\|_{\mathbb{H}} + \left\|A^{\frac{1}{2}}w_t((n-1)\omega)\ \right\|_{\mathbb{H}}$$

$$+\|f((n-1)\omega)\|_{\mathbb{H}} + \omega \max_{t\in[(n-1)\omega,n\omega]} \|f_t\|_{\mathbb{H}} + \max_{(n-1)\omega\leq t\leq n\omega}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H$$

$$+M_3(q,l) \max_{(n-1)\omega\leq t\leq n\omega}\left|\zeta''\right| + M_3(q,l)\int_{(n-1)\omega}^{t}\|w_{zz}(z)\|_H\,dz.$$

Using the integral inequality, we get

$$\max_{(n-1)\omega\leq t\leq n\omega}\|w_{tt}\|_H \leq\ M(q,l)\left[a_n\right.$$

$$\left.+\max_{(n-1)\omega\leq t\leq n\omega}\left\|f'(t)\right\|_H + \|f((n-1)\omega)\|_H + \max_{(n-1)\omega\leq t\leq n\omega}\left|\zeta''\right|\right].$$

In the same manner, we can obtain

$$\max_{(n-1)\omega\leq t\leq n\omega}\left\|A^{\frac{1}{2}}w_t\right\|_H \leq\ M(q,l)\left[a_n\right.$$

$$\left.+\max_{(n-1)\omega\leq t\leq n\omega}\left\|f'(t)\right\|_H + \|f((n-1)\omega)\|_H + \max_{(n-1)\omega\leq t\leq n\omega}\left|\zeta''\right|\right].$$

From that and equation (18) it follows estimate for $\max\limits_{(n-1)\omega\leq t\leq n\omega}\|Aw\|_H$. Theorem 3 is established. □

4. Numerical results

In this section, we study the numerical solution of the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)\sin x + 0.01u_{xx}(t-\pi, x) \\ -1.01\sin t \sin x, \quad t > 0, 0 < x < \pi, \\ u(t,x) = \sin t \sin x, -\pi \leq t \leq 0, \quad 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \int\limits_0^\pi u(t,x)dx = 2\sin t, \quad t \geq 0 \end{cases} \tag{24}$$

for a one dimensional delay hyperbolic differential equation with Dirichlet condition. Note that

$$(u(t,x), p(t)) = ((mu(t,x), mp(t)))_{m=1}^{\infty},$$

where $(mu(t,x), mp(t))$ is exact solution pair of the problem (24) on $t \in [(m-1)\pi, m\pi]$, $m \geq 1$. The exact solution pair of the problem (24) is $(u(t,x), p(t)) = (\sin(t)\sin(x), \sin(t))$. For the numerical solution of problem (24), we present the following first order of accuracy difference scheme for the approximate solution for the problem (24)

$$\begin{cases} \frac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\frac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ =mp_k\sin(x_n)-\sin(t_{k+1})\sin\left(x_n\right),\ m=1, \\ 1\leq k\leq N-1, 1\leq n\leq M-1, \\ \frac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\frac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ =mp_k\sin(x_n)-1.01\sin(t_{k+1})\sin\left(x_n\right), \\ +0.01\frac{(m-1)u_{n+1}^{k-N}-2((m-1)u)_n^{k-N}+(m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k=k\tau,\ x_n=nh, \\ \left(m-1\right)N+1\leq k\leq mN-1, \\ 1\leq n\leq M-1, N\tau=\pi,\ Mh=\pi, m=2,3,..., \\ mu_n^{(m-1)N}=0, \frac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=\sin(x_n), \\ 0\leq n\leq M,\ m=1, \\ mu_n^{(m-1)N}=\left(m-1\right)u_n^{(m-1)N}, \\ \frac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=\frac{(m-1)u_n^{(m-1)N}-(m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0\leq n\leq M,\ m\geq 2, \\ mu_0^{k+1}=mu_M^{k+1}=0, \sum_{i=1}^{M-1}mu_i^{k+1}h=2\sin(t_{k+1}), \\ \left(m-1\right)N\leq k\leq mN,\ m=1,2,.... \end{cases} \tag{25}$$

We consider two cases: $m=1$ and $m\geq 2$. First, let $m=1$, then $0\leq k\leq N$.

From problem (25) it follows that

$$\begin{cases} \frac{1u_n^{k+1}-2(1u)_n^k+1u_n^{k-1}}{\tau^2}-\frac{1u_{n+1}^{k+1}-2(1u)_n^{k+1}+1u_{n-1}^{k+1}}{h^2} \\ = 1p_k\sin(x_n)-\sin(t_{k+1})\sin(x_n), \\ 1\le k\le N-1, 1\le n\le M-1, N\tau=\pi,\ Mh=\pi, \\ 1u_n^0=0, \frac{1u_n^1-1u_n^0}{\tau}=\sin(x_n),\ 0\le n\le M, \\ 1u_0^{k+1}=1u_M^{k+1}=0, \sum_{i=1}^{M-1}1u_i^{k+1}h=2\sin(t_{k+1}), \\ 0\le k\le N. \end{cases} \tag{26}$$

The algorithm for obtaining the solution of the time-dependent identification problem (26) $\{1u_k\}_{k=0}^N=\left\{\{1u_n^k\}_{k=0}^N\right\}_{n=0}^M$ and $\{1p_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$1u_n^k=1\omega_n^k+1\eta_k\sin(x_n),\quad 0\le k\le N,\ 0\le n\le M. \tag{27}$$

Applying difference scheme (26) and formula (27), we obtain formula

$$1\eta_{k+1}=\frac{2\sin(t_{k+1})-\sum_{i=1}^{M-1}1\omega_i^{k+1}h}{\sum_{i=1}^{M-1}\sin(x_i)h},\quad -1\le k\le N-1 \tag{28}$$

and the difference scheme

$$\begin{cases} \frac{1\omega_n^{k+1}-2(1\omega)_n^k+1\omega_n^{k-1}}{\tau^2}-\frac{1\omega_{n+1}^{k+1}-2(1\omega)_n^{k+1}+1\omega_{n-1}^{k+1}}{h^2} \\ +\frac{\sum_{i=1}^{M-1}1\omega_i^{k+1}h}{\sum_{i=1}^{M-1}\sin(x_i)h}\sin(x_n)\frac{2(\cos(h)-1)}{h^2} \\ =\left[\frac{2}{\sum_{i=1}^{M-1}\sin(x_i)h}\frac{2(\cos(h)-1)}{h^2}-1\right]\sin(t_{k+1})\sin(x_n), \\ t_k=k\tau,\ x_n=nh,\ 1\le k\le N-1,\ 1\le n\le M-1, \\ 1\omega_n^0=0, \frac{1\omega_n^1-1\omega_n^0}{\tau}=\sin(x_n),\quad 0\le n\le M, \\ 1\omega_0^{k+1}=1\omega_M^{k+1}=0,\ -1\le k\le N-1. \end{cases} \tag{29}$$

In the first stage, we find numerical solution $\left\{\left\{1\omega_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (29). For obtaining the solution of difference scheme (29), we will write it in the matrix form as

$$\begin{cases} A\,(1\omega)^{k+1} + B\,(1\omega)^{k} + C\,(1\omega)^{k-1} = (1f)^{k}, \\ 1 \le k \le N-1, \\ 1\omega^0 = 0, 1\omega^1 = \tau\sin(x_n), \end{cases} \tag{30}$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $1\omega^s, s = k, k\pm 1, 1f^k$ are $(M+1)\times 1$ column matrices and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & a+\frac{c_1}{d} & b+\frac{c_1}{d} & \cdots & \frac{c_1}{d} & \frac{c_1}{d} & 0 \\ 0 & b+\frac{c_2}{d} & a+\frac{c_2}{d} & \cdots & \frac{c_2}{d} & \frac{c_2}{d} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{c_{M-2}}{d} & \frac{c_{M-2}}{d} & \cdots & a+\frac{c_{M-2}}{d} & b+\frac{c_{M-2}}{d} & 0 \\ 0 & \frac{c_{M-1}}{d} & \frac{c_{M-1}}{d} & \cdots & b+\frac{c_{M-1}}{d} & a+\frac{c_{M-1}}{d} & b \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & g & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & g & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$1f^k = \begin{bmatrix} 0 \\ 1f(t_k, x_1) \\ . \\ 1f(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$1\omega^s = \begin{bmatrix} 1\omega_0^s \\ 1\omega_1^s \\ . \\ 1\omega_{M-1}^s \\ 1\omega_M^s \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1.$$

Here, $a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, c_n = \sin(x_n)\frac{2(\cos(h)-1)}{h}$,

$d = \sum_{i=1}^{M-1} \sin(x_i)h, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2}$,

$1f(t_k, x_n) = \left[\frac{2}{\sum_{i=1}^{M-1} \sin(x_i)h} \frac{2(\cos(h)-1)}{h^2} - 1\right] \sin(t_{k+1}) \sin(x_n)$,

$1 \leq k \leq N-1,\ 1 \leq n \leq M-1.$

So, we have the initial value problem for the second order difference equation (30) with respect to k with matrix coefficients A, B and C: Since ω^0 and ω^1 are given, we can obtain the solution of (30) by direct formula

$$1\omega^{k+1} = A^{-1}(1f^k - B\,(1\omega)^k - C\,(1\omega)^{k-1}),\ k = 1, ..., N-1. \tag{31}$$

Applying formula $1\eta_{k+1} = \sum_{i=1}^{k}(k+1-i)\,(1p)_i\,\tau^2, 1 \leq k \leq N-1, \eta_0 = \eta_1 = 0$, we can obtain

$$1p_k = \frac{1\eta_{k+1} - 2\,(1\eta)_k + 1\eta_{k-1}}{\tau^2},\ 1 \leq k \leq N-1. \tag{32}$$

In the second stage, we will obtain $\{1p_k\}_{k=1}^{N-1}$ by formulas (28) and (32). Finally, in the third stage, we will obtain $\left\{\{1u_n^k\}_{k=0}^{N}\right\}_{n=0}^{M}$ by formulas (27) and (28).

Second, let $m \geq 2$,then $(m-1)\,N \leq k \leq mN$. From problem (25) it follows that

$$\begin{cases} \frac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2} - \frac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ = mp_k \sin(x_n) - 1.01 \sin(t_{k+1}) \sin(x_n), \\ +0.01 \frac{(m-1)u_{n+1}^{k-N}-2((m-1)u)_n^{k-N}+(m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k = k\tau,\ x_n = nh, \\ (m-1)N + 1 \le k \le mN - 1, \\ 1 \le n \le M-1, N\tau = \pi, \quad Mh = \pi, \\ mu_n^{(m-1)N} = (m-1)\, u_n^{(m-1)N}, \\ \frac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau} = \frac{(m-1)u_n^{(m-1)N}-(m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ mu_0^{k+1} = mu_M^{k+1} = 0, \sum_{i=1}^{M-1} mu_i^{k+1} h = 2\sin(t_{k+1}), \\ (m-1)N \le k \le mN,\ m \ge 2. \end{cases} \tag{33}$$

In the same manner, algorithm for obtaining the solution of the time-dependent identification problem (25)

$\{mu_k\}_{k=0}^N = \left\{ \{mu_n^k\}_{k=0}^N \right\}_{n=0}^M$ and $\{mp_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$\begin{cases} mu_n^k = m\omega_n^k + m\eta_k \sin(x_n), \\ (m-1)N \le k \le mN,\ 0 \le n \le M. \end{cases} \tag{34}$$

Applying difference scheme (33) and formula (34), we obtain the formula

$$\begin{cases} m\eta_{k+1} = \frac{2\sin(t_{k+1}) - \sum_{i=1}^{M-1} m\omega_i^{k+1} h}{\sum_{i=1}^{M-1} \sin(x_i) h}, \\ (m-1)N - 1 \le k \le mN - 1, \end{cases} \tag{35}$$

and the difference scheme

$$\begin{cases} \frac{m\omega_n^{k+1}-2(m\omega)_n^k+m\omega_n^{k-1}}{\tau^2} - \frac{m\omega_{n+1}^{k+1}-2(m\omega)_n^{k+1}+m\omega_{n-1}^{k+1}}{h^2} \\ +\frac{\sum_{i=1}^{M-1} m\omega_i^{k+1}h}{\sum_{i=1}^{M-1}\sin(x_i)h}\sin(x_n)\frac{2(\cos(h)-1)}{h^2} \\ = 0.01\frac{((m-1)w)_{n+1}^{k-N}-2((m-1)w)_n^{k-N}+((m-1)w)_{n-1}^{k-N}}{h^2} \\ +\left[\frac{2}{\sum_{i=1}^{M-1}\sin(x_i)h}\frac{2(\cos(h)-1)}{h^2} - 1.01\right]\sin(t_{k+1})\sin(x_n), \\ (m-1)N+1 \le k \le mN-1, \\ m\omega_n^{(m-1)N} = (m-1)\,\omega_n^{(m-1)N}, \\ \frac{m\omega_n^{(m-1)N+1}-m\omega_n^{(m-1)N}}{\tau} = \frac{(m-1)\omega_n^{(m-1)N}-(m-1)\omega_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ m\omega_0^{k+1} = m\omega_M^{k+1} = 0,\ (m-1)N \le k \le mN,\ m \ge 2. \end{cases} \tag{36}$$

In the first stage, we find numerical solution $\left\{\{m\omega_n^k\}_{k=0}^N\right\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (36). For obtaining the solution of difference scheme (36), we will write it in the matrix form as

$$\begin{cases} A\,(m\omega)^{k+1} + B\,(m\omega)^k + C\,(m\omega)^{k-1} = (mf)^k, \\ (m-1)N+1 \le k \le mN-1, \\ (m\omega)_n^{(m-1)N} = ((m-1)\,\omega)_n^{(m-1)N}, \\ (m\omega)_n^{(m-1)N+1} \\ = 2((m-1)\,\omega)_n^{(m-1)N} - ((m-1)\,\omega)_n^{(m-1)N-1}, \end{cases} \tag{37}$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $m\omega^s, s = k, k\pm 1, mf^k$

are $(M+1)\times 1$ column matrices and

$$mf^{k}=\begin{bmatrix} 0 \\ mf(t_{k},x_{1}) \\ . \\ mf(t_{k},x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$m\omega^{s}=\begin{bmatrix} m\omega_{0}^{s} \\ m\omega_{1}^{s} \\ . \\ m\omega_{M-1}^{s} \\ m\omega_{M}^{s} \end{bmatrix}_{(M+1)\times 1}, \text{for } s=k,k\pm 1.$$

So, we have the initial value problem for the second order difference equation (37) with respect to k with matrix coefficients A,B and C: Since $m\omega_{n}^{N}$ and $m\omega_{n}^{N+1}$ are given, we can obtain the solution of (37) by direct formula

$$\begin{cases} (m\omega)^{k+1}=A^{-1}((mf)^{k}-B\,(m\omega)^{k}-C\,(m\omega)^{k-1}), \\ (m-1)\,N+1\leq k\leq mN-1. \end{cases} \tag{38}$$

Applying formula $m\eta_{k+1}=\sum_{i=1}^{k}(k+1-i)\,(mp)_{i}\,\tau^{2},(m-1)\,N+1\leq k\leq mN-1, m\eta_{(m-1)N}=m\eta_{(m-1)N+1}=0$, we can obtain

$$\begin{cases} mp_{k}=\frac{m\eta_{k+1}-2(m\eta)_{k}+m\eta_{k-1}}{\tau^{2}}, \\ (m-1)\,N+1\leq k\leq mN-1. \end{cases} \tag{39}$$

In the second stage, we will obtain $\{mp_{k}\}_{k=1}^{N-1}$ by formulas (35) and (39). Finally, in the third stage, we will obtain $\left\{\{mu_{n}^{k}\}_{k=0}^{N}\right\}_{n=0}^{M}$ by formulas (34) and (35). The errors are computed by

$$mE_{u}=\max_{(m-1)N\leq k\leq mN}\left(\sum_{n=1}^{M-1}\left|u(t_{k},x_{n})-mu_{n}^{k}\right|^{2}h\right)^{\frac{1}{2}}, \tag{40}$$

$$mE_{p}=\max_{(m-1)N+1\leq k\leq mN-1}\left|p(t_{k})-mp_{k}\right|,$$

where $u(t,x),p(t)$ represent the exact solution, mu_{n}^{k} represent the numerical solutions at (t_{k},x_{n}) and mp_{k} represent the numerical solutions at t_{k}. The numerical results are given in the following table.

Table 1. Error analysis

$Error$	$N = M = 40$	$N = M = 80$	$N = M = 160$
$1E_u$	0.0669	0.0345	0.0176
$1E_p$	0.0785	0.0393	0.0196
$2E_u$	0.1655	0.0883	0.0456
$2E_p$	0.0747	0.0379	0.0190
$3E_u$	0.2567	0.1408	0.0739
$3E_p$	0.1418	0.1027	0.0830

As it is seen in Table 1, if M and N are multiplied by 2, the value of errors decreases approximately 1/2 for the DS. This shows that it has the first order of accuracy.

5. Conclusions

In the present paper, a time-dependent source identification problem for a one dimensional delay hyperbolic equation is studied. Operator-functions generated by the positive operator are considered. The main theorems on stability estimates for the solution of the problem (1) are established. The first order of accuracy difference scheme for the source identification problem (1) is presented. Numerical analysis and discussions are presented.

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