

ON THE CONVOLUTION AND NEUTRIX CONVOLUTION
INVOLVING THE FUNCTIONS $\cosh^{-1} x$, $\sinh^{-1} x$ AND x^r

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Abstract: The domain of the function $\cosh^{-1} x$ is extended to the real line by putting $\cosh^{-1} x = 0$ on the interval $(-1, 1)$. The neutrix convolution $\cosh^{-1} x \circledast x^r$ and $\sinh^{-1} x \circledast x^r$ are evaluated for $r = 0, 1, 2, \dots$. Further results are also given.

AMS Subject Classification: 33B15, 33B20, 46F10

Key Words: generalized functions, neutrix, convolution

1. Introduction

The domain of the function $\cosh^{-1} x$ is extended to the real line by putting $\cosh^{-1} x = 0$ on the interval $(-1, 1)$. The functions $\cosh^{-1} x_+$, $\cosh^{-1} x_-$, $\sinh^{-1} x_+$ and $\sinh^{-1} x_-$ are then defined by

$$\cosh^{-1} x_+ = H(x) \cosh^{-1} x, \quad \cosh^{-1} x_- = H(-x) \cosh^{-1} x,$$

$$\sinh^{-1} x_+ = H(x) \sinh^{-1} x, \quad \sinh^{-1} x_- = H(-x) \sinh^{-1} x,$$

where H denotes Heaviside's function. Note that

$$\cosh^{-1} x = \cosh^{-1} x_+ + \cosh^{-1} x_-$$

and

$$\sinh^{-1} x = \sinh^{-1} x_+ + \sinh^{-1} x_-.$$

The classical definition of the convolution of two functions f and g is as follows:

Definition 1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt \quad (1)$$

for all points x for which the integral exists.

It follows from Definition 1 that if $f * g$ exists, then $g * f$ exists and

$$f * g = g * f. \quad (2)$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \quad (3)$$

Gel'fand and Shilov [5] extended Definition 1 to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$, the space of infinitely differentiable functions with compact support.

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle \quad (4)$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the following conditions:

- (a) either f or g has bounded support,
- (b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition, then equations (2) and (3) are satisfied.

The definition of the convolution is rather restrictive and so the non-commutative neutrix convolution was introduced in [2]. In order to define the neutrix convolution we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
- (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n \end{cases}$$

for $n = 1, 2, \dots$.

We have the following definition of the commutative neutrix convolution product.

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \dots$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense

$$N\text{-}\lim_{n \rightarrow \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' , the real numbers, with negligible functions being finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, if

$$\lim_{n \rightarrow \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, we say that the *convolution* $f * g$ exists and equals h .

Note that in this definition the convolution $f_n * g$ is as defined in Gel'fand and Shilov's sense, the distribution f_n having compact support. Note also that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

We need the following results which are proved in [3] and [4]:

$$\begin{aligned}\cosh^{2r} x &= 2^{1-2r} \sum_{k=1}^r \binom{2r}{r-k} \cosh(2kx) + 2^{-2r} \binom{2r}{r}, \\ \cosh^{2r-1} x &= 2^{2-2r} \sum_{k=1}^r \binom{2r-1}{r-k} \cosh(2k-1)x,\end{aligned}$$

for $r = 1, 2, \dots$. For shortness, we will write

$$\cosh^r x = \sum_{k=0}^r a_{r,k} \cosh(kx),$$

where $a_{2r,2k+1} = a_{2r-1,2k} = 0$, for $k = 1, 2, \dots, r$, so that

$$\begin{aligned}\cosh^{2r} x &= \sum_{k=0}^r a_{2r,2k} \cosh(2kx), \\ \cosh^{2r-1} x &= \sum_{k=1}^r a_{2r-1,2k-1} \cosh(2k-1)x,\end{aligned}$$

$$\begin{aligned}\sinh(2rx) &= \sum_{k=1}^r \sum_{j=0}^{k-1} \binom{2r}{2k-1} \binom{k-1}{j} (-1)^{k+j+1} \cosh^{2r-2k+2j+1} x \sinh x, \\ \sinh(2r-1)x &= \sum_{k=1}^r \sum_{j=0}^{k-1} \binom{2r-1}{2k-1} \binom{k-1}{j} (-1)^{k+j+1} \cosh^{2r-2k+2j} x \sinh x,\end{aligned}$$

for $r = 1, 2, \dots$. For shortness, we will write

$$\sinh(rx) = \sum_{k=1}^r b_{r,k} \cosh^k x \sinh x,$$

for $r = 1, 2, \dots$ and $k = 1, 2, \dots, r$, where $b_{2r,2k} = b_{2r-1,2k-1} = 0$, for $k = 1, 2, \dots, r$, so that

$$\begin{aligned}\sinh(2rx) &= \sum_{k=1}^r b_{2r,2k+1} \cosh^{2r-2k+1} x \sinh x, \\ \sinh(2r-1)x &= \sum_{k=1}^r b_{2r-1,2k} \cosh^{2r-2k} x \sinh x. \\ \int \cosh^r x \, dx &= \sum_{k=1}^r \sum_{i=1}^k \frac{a_{r,k} b_{k,i}}{k} \cosh^i x \sinh x + a_{r,0} x, \quad (5)\end{aligned}$$

for $r = 1, 2, \dots$.

$$\begin{aligned}\cosh^{-1} x_+ * x_+^r &= H(x-1) \sum_{k=0}^r \binom{r}{k} \left[\frac{(-1)^k (x^{r+1} - a_{k+1,0} x^{r-k}) \cosh^{-1} x}{k+1} \right. \\ &\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{(-1)^k a_{k+1,i} b_{i,j}}{i(k+1)} x^{r+j-k} (x^2 - 1)^{1/2} \right], \quad (6)\end{aligned}$$

$$\cosh^{-1} x_+ \circledast x^r = \sum_{k=0}^r \binom{r}{k} (-1)^k x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \quad (7)$$

where

$$c_k = \begin{cases} 0, & k \text{ odd,} \\ (-1)^{1+k/2} \binom{-1/2}{k/2} \frac{1}{k}, & k \text{ even,} \end{cases} \quad d_k = \begin{cases} 0, & j \text{ even,} \\ \binom{1/2}{(j+1)/2}, & j \text{ odd.} \end{cases}$$

$$\cosh^{-1} x \circledast x^r = \sum_{k=0}^r \binom{r}{k} [(-1)^k + 1] x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \quad (8)$$

for $r = 1, 2, \dots$.

$$\begin{aligned}\sinh^{-1} x_+ * x_+^r &= \sum_{k=0}^r \binom{r}{k} \left[\frac{(-1)^k x_+^{r+1} \sinh^{-1} x}{k+1} \right. \\ &\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2 + 1)^{j/2} \right], \quad (9)\end{aligned}$$

$$\sinh^{-1} x_+ \circledast x^r = \sum_{k=0}^r \binom{r}{k} (-1)^k x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \quad (10)$$

$$\sinh^{-1} x \circledast x^r = \sum_{k=0}^r \binom{r}{k} [(-1)^k - 1] x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right] \quad (11)$$

for $r = 1, 2, \dots$.

We now prove the following theorem.

Theorem 3. *The neutrix convolution $x^r \circledast \cosh^{-1} x_+$ exists and*

$$\begin{aligned} x^r \circledast \cosh^{-1} x_+ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} + \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+i+1} x^{r+1-m}}{2i} + \\ &\left. + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{k+l+1} x^{r-k+j+1-2l} \right] \quad (12) \end{aligned}$$

for $r = 0, 1, 2, \dots$, where

$$i_k = \begin{cases} \frac{k}{2}, & k \text{ even,} \\ \frac{k-1}{2}, & k \text{ odd} \end{cases} \quad l_j = \begin{cases} \frac{j}{2}, & j \text{ even,} \\ \frac{j+1}{2}, & j \text{ odd.} \end{cases}$$

Proof. Putting $[x^r]_n = x^r \tau_n(x)$, we have

$$\begin{aligned} [x^r]_n * \cosh^{-1} x_+ &= \int_1^{x+n} \cosh^{-1} t (x-t)^r dt \\ &+ \int_{x+n}^{x+n+n^{-n}} \cosh^{-1} t (x-t)^r \tau_n(t) dt \\ &= \sum_{k=0}^r \binom{r}{k} (-1)^k x^{r-k} \int_1^{x+n} t^k \cosh^{-1} t dt \\ &+ \int_{x+n}^{x+n+n^{-n}} \cosh^{-1} t (x-t)^r \tau_n(t) dt \\ &= I_1 + I_2. \end{aligned} \quad (13)$$

Making the substitution $t = \cosh u$, we get

$$\begin{aligned}
 \int_1^{x+n} t^k \cosh^{-1} t \, dt &= \int_0^{\cosh^{-1}(x+n)} u \cosh^k u \sinh u \, du \\
 &= \frac{(x+n)^{k+1} \cosh^{-1}(x+n)}{k+1} - \int_0^{\cosh^{-1}(x+n)} \frac{\cosh^{k+1} u}{k+1} \, du \\
 &= \frac{((x+n)^{k+1} - a_{k+1,0}) \cosh^{-1}(x+n)}{k+1} \\
 &\quad - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} (x+n)^j ((x+n)^2 - 1)^{1/2}. \quad (14)
 \end{aligned}$$

Now, when $x > 1$, we have

$$[\cosh^{-1}(x+n)]' = ((x+n)^2 - 1)^{-1/2} = (x+n)^{-1} \sum_{i=0}^{\infty} \binom{-1/2}{i} (-1)^i (x+n)^{-2i},$$

and so

$$\begin{aligned}
 \cosh^{-1}(x+n) &= \ln(x+n) - \sum_{i=1}^{\infty} \binom{-1/2}{i} \frac{(-1)^i (x+n)^{-2i}}{2i} + \text{constant} \\
 &= \ln n + \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^i}{i n^i} \\
 &\quad + \sum_{i=1}^{\infty} \binom{-1/2}{i} \frac{(-1)^{i+1} (x+n)^{-2i}}{2i} + \text{constant}.
 \end{aligned}$$

Hence, for $k = 1, 2, \dots$, we have

$$\begin{aligned}
 N\text{-}\lim_{n \rightarrow \infty} (x+n)^k \cosh^{-1}(x+n) &= \sum_{m=1}^k \binom{k}{m} \frac{(-1)^{m+1}}{m} x^k + \\
 &\quad + \sum_{m=2i}^{2i_k} \sum_{i=1}^{i_k} \binom{k}{m} \binom{-1/2}{i} \frac{(-1)^{i+1} x^{k-m}}{2i} \quad (15)
 \end{aligned}$$

where

$$i_k = \begin{cases} \frac{k}{2}, & k \text{ even,} \\ \frac{k-1}{2}, & k \text{ odd.} \end{cases} \quad (16)$$

Further,

$$((x+n)^2 - 1)^{1/2} = (x+n) \sum_{l=0}^{\infty} \binom{1/2}{l} (-1)^l (x+n)^{-2l},$$

and so

$$\text{N-lim}_{n \rightarrow \infty} (x+n)^j ((x+n)^2 - 1)^{1/2} = \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^l x^{j+1-2l}, \quad (17)$$

where

$$l_j = \begin{cases} \frac{j}{2}, & j \text{ even,} \\ \frac{j+1}{2}, & j \text{ odd.} \end{cases} \quad (18)$$

Hence

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} I_1 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+i+1} x^{r+1-m}}{2i} \\ &\left. + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{k+l+1} x^{r-k+j+1-2l} \right]. \quad (19) \end{aligned}$$

Next, it is easily seen that $I_2 = O(n^{-n})$ and so

$$\lim_{n \rightarrow \infty} I_2 = 0. \quad (20)$$

Equation (12) now follows from equations (13), (19) and (20).

Replacing x by $-x$ in equation (12), we get

Corollary 3.1 *The neutrix convolution $x^r \circledast \cosh^{-1} x_-$ exists and*

$$\begin{aligned} x^r \circledast \cosh^{-1} x_- &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m}}{m} x^{r+1} \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+i} x^{r+1-m}}{2i} \\ &\left. + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{j+l} x^{r-k+j+1-2l} \right], \quad (21) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Corollary 3.2 *The neutrix convolution $x^r \circledast \cosh^{-1} x$ exists and*

$$\begin{aligned} x^r \circledast \cosh^{-1} x &= \sum_{k=0}^r \sum_{i=1}^{k+1} \sum_{j=1}^i \binom{r}{k} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} \\ &\quad \times \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r-k+j+1-2l} [(-1)^{k+l+1} + (-1)^{j+l}], \quad (22) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Proof. We have

$$x^r \circledast \cosh^{-1} x = x^r \circledast \cosh^{-1} x_+ + x^r \circledast \cosh^{-1} x_-,$$

and then equation (22) follows from equations (12) and (21).

Corollary 3.3. *The neutrix convolution $x_-^r \circledast \cosh^{-1} x_+$ exists and*

$$\begin{aligned} x_-^r \circledast \cosh^{-1} x_+ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{r+k+m+1}}{m} x^{r+1} + \right. \\ &\quad + \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{r+k+i+1} x^{r+1-m}}{2i} + \\ &\quad + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{r+k+l+1} x^{r-k+j+1-2l} \\ &\quad + (-1)^{r+1+k} H(x-1) \left[\frac{(x^{r+1} - a_{k+1,0} x^{r-k}) \cosh^{-1} x}{k+1} \right. \\ &\quad \left. \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} (-1)^{j+k+1}}{i(k+1)} x^{r+j-k} (x^2 - 1)^{1/2} \right] \right] \quad (23) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Proof. We have

$$\begin{aligned}
 (-1)^r x_-^r \circledast \cosh^{-1} x_+ &= x^r \circledast \cosh^{-1} x_+ - x_+^r * \cosh^{-1} x_+ \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} + \right. \\
 &\quad + \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+i+1} x^{r+1-m}}{2i} + \\
 &\quad + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{k+l+1} x^{r-k+j+1-2l} \Big] \\
 &\quad - H(x-1) \sum_{k=0}^r \binom{r}{k} \left[\frac{(-1)^k (x^{r+1} - a_{k+1,0} x^{r-k}) \cosh^{-1} x}{k+1} \right. \\
 &\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{(-1)^k a_{k+1,i} b_{i,j}}{i(k+1)} x^{r+j-k} (x^2 - 1)^{1/2} \right].
 \end{aligned}$$

By using equations (6) and (12) equation (23) follows.

Replacing x by $-x$ in equation (23), we get

Corollary 3.4. *The neutrix convolution $x_+^r \circledast \cosh^{-1} x_-$ exists and*

$$\begin{aligned}
 x_+^r \circledast \cosh^{-1} x_- &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m}}{m} x^{r+1} + \right. \\
 &\quad + \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+i} x^{k+1-m}}{2i} + \\
 &\quad + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} \sum_{l=0}^{l_j} \binom{1/2}{l} (-1)^{j+l} x^{r-k+j+1-2l} \Big] \\
 &\quad + (-1)^k H(x-1) \sum_{k=0}^r \binom{r}{k} \left[\frac{(x^{r+1} + (-1)^k a_{k+1,0} x^{r-k}) \cosh^{-1} x}{k+1} \right. \\
 &\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x^{r+j-k} (x^2 - 1)^{1/2} \right] \tag{24}
 \end{aligned}$$

for $r = 0, 1, 2, \dots$

Theorem 4. *The neutrix convolution $x^r \circledast \sinh^{-1} x_+$ exists and*

$$\begin{aligned} x^r \circledast \sinh^{-1} x_+ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^k x^{r+1-m}}{2i} \\ &\left. + (-1)^{k+1} \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \right], \quad (25) \end{aligned}$$

for $r = 0, 1, 2, \dots$.

Proof. Putting $[x^r]_n = x^r \tau_n(x)$, we have

$$\begin{aligned} [x^r]_n * \sinh^{-1} x_+ &= \int_0^{x+n} \sinh^{-1} t (x-t)^r dt \\ &+ \int_{x+n}^{x+n+n^{-n}} \sinh^{-1} t (x-t)^r \tau_n(t) dt \\ &= \sum_{k=0}^r \binom{r}{k} (-1)^k x^{r-k} \int_0^{x+n} t^k \sinh^{-1} t dt \\ &+ \int_{x+n}^{x+n+n^{-n}} \sinh^{-1} t (x-t)^r \tau_n(t) dt \\ &= I_1 + I_2. \quad (26) \end{aligned}$$

Making the substitution $t = \sinh u$, we get

$$\begin{aligned} \int_0^{x+n} t^k \sinh^{-1} t dt &= \int_0^{\sinh^{-1}(x+n)} u \sinh^k u \cosh u du \\ &= \frac{(x+n)^{k+1} \sinh^{-1}(x+n)}{k+1} - \int_0^{\sinh^{-1}(x+n)} \frac{\sinh^{k+1} u}{k+1} du \\ &= \frac{(x+n)^{k+1} \sinh^{-1}(x+n)}{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} (x+n) ((x+n)^2 + 1)^{j/2}. \quad (27) \end{aligned}$$

Now,

$$[\sinh^{-1}(x+n)]' = ((x+n)^2 + 1)^{-1/2} = (x+n)^{-1} \sum_{i=0}^{\infty} \binom{-1/2}{i} (x+n)^{-2i}$$

and so

$$\sinh^{-1}(x+n) = \ln(x+n) - \sum_{i=1}^{\infty} \binom{-1/2}{i} \frac{(x+n)^{-2i}}{2i} + \text{constant} \quad (28)$$

$$= \ln n + \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^i}{i n^i} - \sum_{i=1}^{\infty} \binom{-1/2}{i} \frac{(x+n)^{-2i}}{2i} + \text{constant}. \quad (29)$$

Hence, for $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} (x+n)^k \sinh^{-1}(x+n) &= \sum_{m=1}^k \binom{k}{m} \frac{(-1)^{m+1}}{m} x^k + \\ &+ \sum_{m=2i}^{2i_k} \sum_{i=1}^{i_k} \binom{k}{m} \binom{-1/2}{i} \frac{x^{k-m}}{2i}. \end{aligned} \quad (30)$$

Further,

$$((x+n)^2 + 1)^{j/2} = (x+n)^j \sum_{i=0}^{\infty} \binom{j/2}{i} (x+n)^{-2i}$$

and so for $j = 1, 2, \dots$, we have

$$\text{N-lim}_{n \rightarrow \infty} (x+n)((x+n)^2 + 1)^{j/2} = \sum_{l=0}^{l_j} \binom{1/2}{l} x^{j+1-2l}. \quad (31)$$

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} I_1 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} + \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^k x^{r+1-m}}{2i} + \\ &\left. + (-1)^{k+1} \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \right] \end{aligned} \quad (32)$$

Next, it is easily seen that $I_2 = O(n^{-n})$ and so

$$\lim_{n \rightarrow \infty} I_2 = 0. \quad (33)$$

Equation (25) now follows from equations (26), (32) and (33).

Replacing x by $-x$ in equation (25), we get

Corollary 4.1. *The neutrix convolution $x^r \circledast \sinh^{-1} x_-$ exists and*

$$\begin{aligned} x^r \circledast \sinh^{-1} x_- &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m}}{m} x^{r+1} + \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+1} x^{r+1-m}}{2i} + \\ &+ \left. \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} (-1)^j}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \right], \quad (34) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Corollary 4.2. *The neutrix convolution $x^r \circledast \sinh^{-1} x$ exists and*

$$\begin{aligned} x^r \circledast \sinh^{-1} x &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j} (-1)^{r+j}}{i} \right. \\ &\times \left. \sum_{l=0}^{l_j} \binom{1/2}{l} [(-1)^{k+1} + (-1)^j] x^{r+j+1-k-2l} \right] \quad (35) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Proof. We have

$$x^r \circledast \sinh^{-1} x = x^r \circledast \sinh^{-1} x_+ + x^r \circledast \sinh^{-1} x_-$$

and then equation (35) follows from equations (25) and (34).

Corollary 4.3. *The neutrix convolution $x_-^r \circledast \sinh^{-1} x_+$ exists and*

$$\begin{aligned} x_-^r \circledast \sinh^{-1} x_+ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{r+k+m+1}}{m} x^{r+1} \right. \\ &+ \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{r+k} x^{r+1-m}}{2i} + (-1)^{r+k+1} \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \\ &\times \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \left. \right] + (-1)^{r+1} \sum_{k=0}^r \binom{r}{k} \\ &\times \left[\frac{(-1)^k x_+^{r+1} \sinh^{-1} x}{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2 + 1)^{j/2} \right] \quad (36) \end{aligned}$$

for $r = 0, 1, 2, \dots$

Proof. We have

$$\begin{aligned}
 (-1)^r x_-^r \otimes \sinh^{-1} x_+ &= x^r \otimes \sinh^{-1} x_+ - x_+^r * \sinh^{-1} x_+ \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m+1}}{m} x^{r+1} \right. \\
 &\quad + \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^k x^{r+1-m}}{2i} \\
 &\quad + (-1)^{k+1} \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i} \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \Big] \\
 &\quad - \sum_{k=0}^r \binom{r}{k} \left[\frac{(-1)^k x_+^{r+1} \sinh^{-1} x}{k+1} \right. \\
 &\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2 + 1)^{j/2} \right].
 \end{aligned}$$

By using equations (9) and (25) equation (36) follows.

Replace x by $-x$ in equation (36), we get

Corollary 4.4. *The neutrix convolution $x_+^r \otimes \sinh^{-1} x_-$ exists and*

$$\begin{aligned}
 x_+^r \otimes \sinh^{-1} x_- &= \sum_{k=0}^r \binom{r}{k} \frac{1}{k+1} \left[\sum_{m=1}^{k+1} \binom{k+1}{m} \frac{(-1)^{k+m}}{m} x^{r+1} \right. \\
 &\quad + \sum_{m=2i}^{2i_{k+1}} \sum_{i=1}^{i_{k+1}} \binom{k+1}{m} \binom{-1/2}{i} \frac{(-1)^{k+1} x^{r+1-m}}{2i} + \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{(-1)^j a_{k+1,i} b_{i,j}}{i} \\
 &\quad \times \sum_{l=0}^{l_j} \binom{1/2}{l} x^{r+j+1-k-2l} \Big] + (-1)^{r+1} \sum_{k=0}^r \binom{r}{k} \\
 &\quad \times \left[\frac{(-1)^k x_-^{r+1} \sinh^{-1} x}{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_-^{r-k+1} (x^2 + 1)^{j/2} \right] \quad (37)
 \end{aligned}$$

for $r = 0, 1, 2, \dots$

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