

CONVOLUTION THEOREM AND AN ALTERNATIVE TO THE OCTONIONIC FOURIER TRANSFORM DEFINITION

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Abstract: In this paper we present a construction of the Octonionic Fourier Series and we introduce a version for the Octonionic Fourier Transform with hypercomplex exponentials, besides we discuss a possible way of defining the convolution product for octonionic functions and also theoretical results. Through some examples, we illustrate the developed concepts for the octonionic transform and the convolution product.

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1. Introduction and Motivation

Octonions are hypercomplex numbers and can be considered as non-associative quaternions in some ways. They have been applied in larger physics [13], [15], such as M theory, the strings and theories of alternative gravity. In this work, in the future we intend to make a greater application of the hypercomplexes in unified field theories, and motivated both by previous works by Eilenberg, Niven [3], Deavours [14], Sinegre [1], and in recent results obtained by some of the authors [10, 11, 12, 4, 6, 5], Boek, Gurlebeck [?], and Lam [2], we focused on the construction of an extension of Algebra of the hypercomplex, more specifically in the construction of the Fourier transform Octonions.

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The quaternions were introduced in 1843 by William R. Hamilton, at the same time John T. Graves, a friend of Hamilton, found an 8-dimensional algebra whose property in non-associativity in a multiplication table is valid. Two years later, in 1845, after some contributions on the subject by Arthur Cayley, octonions were also called “Cayley numbers”.

In this context of the present work, and based on an earlier result (Pendeza et al., op. cit. [7]), we present for octonions the concept of a periodic function of $2L$, which we will call the Fourier Octonionic Series. We define an exponential octagonal function and show a non-associative expansion of Moivre’s Theorem and generalizations of the Euler formula. Finally, we obtain octonionic versions of the Fourier Transform and present the convolution theorem.

This paper is organized as follows. Section 2 describes the defined Octonionic Fourier Series, based in [7]. Section 3 describes the construction the Octonionic Fourier transform and presents the version of the convolution theorem for octonionics. Final remarks are given in Section 4.

2. Octonionic Fourier Series

The octonions are a somewhat nonassociative extension of the quaternions. They form the 8-dimensional normed division algebra on $\mathcal{R}$.

The octonionic algebra, also called octaves denoted for $\mathcal{O}$, is an alternative division algebra.

The octonions set $\mathcal{O}$ is denoted by

$$\mathcal{O} = \{a, b, c, d, e, f, g, h\} \in \mathcal{R},$$

where,

$$\begin{aligned} (a, b, c, d, e, f, g, h) &= (a', b', c', d', e', f', g', h') \\ \iff a = a', b = b', c = c', d = d', e = e', f = f', g = g', h = h'. \end{aligned}$$

The octonions do not form a ring due the non-commutativity of the multiplication. Also do not form a group due a nonassociative of multiplication. They form a *Moufang Loop*, a Loop with identity element (Jacobson [8]).

Let us consider an octonionic number given by

$$o = a + bi + cj + dk + el + fi + gj + hk.$$

The octonion unities **1**, **i**, **j**, **k**, **l**, **li**, **lj**, **lk** form an orthonormal base of the 8-dimensional algebra.

Similarly to [6] we start our contribution in considering f a function defined on interval $[-L, L]$, $L > 0$, and outside of this interval set as $f(x) = f(x + 2L)$, that is, $f(x)$ is $2L$ -periodical. If f and f' are piecewise continuous then the series of function given below,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad (1)$$

is convergent and the limit is $\tilde{f}(x) = \frac{\lim_{a \rightarrow x^+} f(x) + \lim_{a \rightarrow x^-} f(x)}{2}$.

The coefficients of Fourier of f , a_0 , a_n and b_n are given by:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad (2)$$

and

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (3)$$

The trigonometric series presented in (1) with this choice of coefficients is the Fourier series of f .

Now, we will detail some properties considering octonions $o \in \mathcal{O}$ given by

$$o = u_1 + u_2 i + u_3 j + u_4 k + u_5 l + u_6 li + u_7 lj + u_8 lk = u_1 + \vec{u}.$$

The extended equation of De Moivre for octonions is given by:

$$e^o = e^{u_1} \left\{ \cos \left(\sqrt{u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2} \right) + \vec{u} \left(\frac{\sin(\sqrt{u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2})}{\sqrt{u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2}} \right) \right\}. \quad (4)$$

Then, using (4) we can obtain:

$$\frac{e^{iy} + e^{-iy}}{2} = \frac{e^0(\cos|y| + i\frac{y}{|y|}\sin|y|) + e^0(\cos|-y| + i\frac{-y}{|-y|}\sin|-y|)}{2}.$$

Since $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$ we have that

$$\frac{e^{iy} + e^{-iy}}{2} = \cos(y).$$

Following the same arguments we can get

$$\begin{aligned}\cos y &= \frac{e^{iy} + e^{-iy}}{2} = \frac{e^{jy} + e^{-jy}}{2} = \frac{e^{ky} + e^{-ky}}{2} = \frac{e^{ly} + e^{-ly}}{2} \\ &= \frac{e^{(li)y} + e^{-(li)y}}{2} = \frac{e^{(lj)y} + e^{-(lj)y}}{2} = \frac{e^{(lk)y} + e^{-(lk)y}}{2},\end{aligned}\quad (5)$$

and

$$\begin{aligned}\sin y &= \frac{e^{iy} - e^{-iy}}{2i} = \frac{e^{jy} - e^{-jy}}{2j} = \frac{e^{ky} - e^{-ky}}{2k} = \frac{e^{ly} - e^{-ly}}{2l} \\ &= \frac{e^{(li)y} - e^{-(li)y}}{2li} = \frac{e^{(lj)y} - e^{-(lj)y}}{2lj} = \frac{e^{(lk)y} - e^{-(lk)y}}{2lk}.\end{aligned}\quad (6)$$

From (5) and (6), we obtain,

$$\begin{aligned}\cos y &= \frac{(e^{iy} + e^{jy} + \dots + e^{(lk)y}) + (e^{-iy} + e^{-jy} + \dots + e^{-(lk)y})}{14}, \\ \sin y &= \frac{1}{7} \left[\frac{e^{iy} - e^{-iy}}{2i} + \frac{e^{jy} - e^{-jy}}{2j} + \dots + \frac{e^{(lk)y} - e^{-(lk)y}}{2lk} \right].\end{aligned}$$

Consequently,

$$\begin{aligned}a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{7} \left[\underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{c_n^1} e^{\frac{in\pi x}{L}} + \right. \\ &\quad \left. \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{c_n^1} e^{-\frac{in\pi x}{L}} + \dots + \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2(lk)}\right)}_{c_n^7} e^{\frac{(lk)n\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2(lk)}\right)}_{c_n^7} e^{-\frac{(lk)n\pi x}{L}} \right].\end{aligned}$$

Following the reasoning presented in [7], we can get

$$\begin{aligned}\sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] &= \frac{1}{7} \sum_{-\infty, n \neq 0}^{\infty} \left[c_n^1 e^{\frac{in\pi x}{L}} + c_n^2 e^{\frac{jn\pi x}{L}} \right. \\ &\quad \left. + c_n^3 e^{\frac{kn\pi x}{L}} + c_n^4 e^{\frac{ln\pi x}{L}} + c_n^5 e^{\frac{lin\pi x}{L}} + c_n^6 e^{\frac{ljn\pi x}{L}} + c_n^7 e^{\frac{lk n\pi x}{L}} \right].\end{aligned}$$

Since $f : R \rightarrow R$ is periodic, with period $2L$, f and f' are piecewise continuous, we have that the Fourier series of f presented in (1) can be written as follows:

$$\begin{aligned} \tilde{f}(x) = c_0 + \frac{1}{7} \left\{ \sum_{-\infty, n \neq 0}^{\infty} c_n^1 e^{\frac{in\pi x}{L}} + c_n^2 e^{\frac{jn\pi x}{L}} + c_n^3 e^{\frac{kn\pi x}{L}} + c_n^4 e^{\frac{ln\pi x}{L}} \right. \\ \left. + c_n^5 e^{\frac{lin\pi x}{L}} + c_n^6 e^{\frac{ljn\pi x}{L}} + c_n^7 e^{\frac{lk n\pi x}{L}} \right\}, \end{aligned} \quad (7)$$

where considering $c_0 = \frac{a_0}{2}$.

The series (7) is the octonionic Fourier series of f .

3. The Octonionic Fourier Transform

In this section we will consider the definition of Octonionic Fourier Series (7) to deduce a model for Octonionic Fourier Transform. From equation (7), consider f defined in the interval $(-L, L)$ with f and f' are piecewise continuous. Denoting $\alpha_n = \frac{n\pi}{L}$, we obtain

$$\begin{aligned} \tilde{f}(x) = \frac{1}{2L} \int_{-L}^L f(u) du + \frac{1}{14L} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[\int_{-L}^L f(u) e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} \right. \\ + \left[\int_{-L}^L f(u) e^{-j(\alpha_n u)} du \right] e^{j\alpha_n x} + \left[\int_{-L}^L f(u) e^{-k(\alpha_n u)} du \right] e^{k\alpha_n x} \\ + \left[\int_{-L}^L f(u) e^{-l(\alpha_n u)} du \right] e^{l\alpha_n x} + \left[\int_{-L}^L f(u) e^{-li(\alpha_n u)} du \right] e^{li\alpha_n x} \\ \left. + \left[\int_{-L}^L f(u) e^{-lj(\alpha_n u)} du \right] e^{lj\alpha_n x} + \left[\int_{-L}^L f(u) e^{-lk(\alpha_n u)} du \right] e^{lk\alpha_n x} \right\}, \end{aligned}$$

considering $\Delta\alpha = \alpha_{n+1} - \alpha_n$,

$$\begin{aligned} = \frac{\Delta\alpha}{2\pi} \int_{-L}^L f(u) du + \frac{1}{14\pi} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[\int_{-L}^L f(u) e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} \right. \\ + \left[\int_{-L}^L f(u) e^{-j(\alpha_n u)} du \right] e^{j\alpha_n x} + \left[\int_{-L}^L f(u) e^{-k(\alpha_n u)} du \right] e^{k\alpha_n x} \\ + \left[\int_{-L}^L f(u) e^{-l(\alpha_n u)} du \right] e^{l\alpha_n x} + \left[\int_{-L}^L f(u) e^{-li(\alpha_n u)} du \right] e^{li\alpha_n x} \\ \left. + \left[\int_{-L}^L f(u) e^{-lj(\alpha_n u)} du \right] e^{lj\alpha_n x} + \left[\int_{-L}^L f(u) e^{-lk(\alpha_n u)} du \right] e^{lk\alpha_n x} \right\} \Delta\alpha, \end{aligned}$$

calculating the limit $\Delta\alpha \rightarrow 0$ and assuming $\int_{-\infty}^{\infty} |f(u)|du < \infty$,

$$= \lim_{\Delta\alpha \rightarrow 0} \frac{1}{14\pi} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[\int_{-L}^L f(u) e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} + \dots \right. \\ \left. + \left[\int_{-L}^L f(u) e^{-lj(\alpha_n u)} du \right] e^{lj\alpha_n x} + \left[\int_{-L}^L f(u) e^{-lk(\alpha_n u)} du \right] e^{lk\alpha_n x} \right\} \Delta\alpha.$$

From the definition of the Riemann integral, we obtain

$$= \frac{1}{14\pi} \int_{-\infty}^{\infty} \left\{ \left[\int_{-\infty}^{\infty} f(u) e^{-i(\alpha u)} du \right] e^{i\alpha x} + \left[\int_{-\infty}^{\infty} f(u) e^{-j(\alpha u)} du \right] e^{j\alpha x} + \dots \right. \\ \left. + \left[\int_{-\infty}^{\infty} f(u) e^{-lj(\alpha u)} du \right] e^{lj\alpha x} + \left[\int_{-\infty}^{\infty} f(u) e^{-lk(\alpha u)} du \right] e^{lk\alpha x} \right\} d\alpha.$$

Or equivalently, we get

$$\frac{1}{14\pi} \int_{-\infty}^{\infty} \left\langle \left(\int_{-\infty}^{\infty} f(u) e^{-i(\alpha u)} du, \int_{-\infty}^{\infty} f(u) e^{-j(\alpha u)} du, \int_{-\infty}^{\infty} f(u) e^{-k(\alpha u)} du, \right. \right. \\ \left. \int_{-\infty}^{\infty} f(u) e^{-l(\alpha u)} du, \int_{-\infty}^{\infty} f(u) e^{-li(\alpha u)} du, \int_{-\infty}^{\infty} f(u) e^{-lj(\alpha u)} du, \right. \\ \left. \left. \int_{-\infty}^{\infty} f(u) e^{-lk(\alpha u)} du \right) (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}, e^{l\alpha x}, e^{li\alpha x}, e^{lj\alpha x}, e^{lk\alpha x}) \right\rangle d\alpha, \quad (8)$$

considering the notations

$$\begin{aligned} \mathfrak{T}_i &= F_i(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i(\alpha u)} du, \\ \mathfrak{T}_j &= F_j(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-j(\alpha u)} du, \\ \mathfrak{T}_k &= F_k(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-k(\alpha u)} du, \\ \mathfrak{T}_l &= F_l(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-l(\alpha u)} du, \\ \mathfrak{T}_{li} &= F_{li}(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-li(\alpha u)} du, \\ \mathfrak{T}_{lj} &= F_{lj}(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-lj(\alpha u)} du, \\ \mathfrak{T}_{lk} &= F_{lk}(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-lk(\alpha u)} du, \end{aligned}$$

the equation (8) can be written as

$$\begin{aligned} \tilde{f}(x) = \frac{1}{14\pi} \int_{-\infty}^{\infty} \langle (F_i(\alpha), F_j(\alpha), F_k(\alpha), F_l(\alpha), F_{li}(\alpha), F_{lj}(\alpha), F_{lk}(\alpha)), \\ (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}, e^{l\alpha x}, e^{li\alpha x}, e^{lj\alpha x}, e^{lk\alpha x}) \rangle d\alpha. \end{aligned} \quad (9)$$

We define Octonionic Fourier Transform as

$$F_{\mathbb{O}}(\alpha) = \mathfrak{T}_{\mathbb{O}}\{f(x)\} = (F_i(\alpha), F_j(\alpha), F_k(\alpha), F_l(\alpha), F_{li}(\alpha), F_{lj}(\alpha), F_{lk}(\alpha)).$$

The inverse transform is calculated by the equation

$$\mathfrak{T}_{\mathbb{O}}^{-1}\{F_{\mathbb{O}}(\alpha)\} = \frac{1}{14\pi} \int_{-\infty}^{\infty} \langle F_{\mathbb{O}}(\alpha), (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}, e^{l\alpha x}, e^{li\alpha x}, e^{lj\alpha x}, e^{lk\alpha x}) \rangle d\alpha.$$

Now we present below a numerical example for the Octonionic Fourier Transform calculation.

Example 1. Consider the function

$$f(x) = \begin{cases} 1, & \text{for } -\pi < x < 0, \\ -1, & \text{for } 0 < x < \pi. \end{cases}$$

Computing the Octonionic Fourier Transform we obtain

$$\begin{aligned} F_{\mathbb{O}}(\alpha) &= \mathfrak{T}_{\mathbb{O}}\{f(x)\} \\ &= 2 \left(i \frac{\cos(\alpha\pi) - 1}{\alpha}, j \frac{\cos(\alpha\pi) - 1}{\alpha}, k \frac{\cos(\alpha\pi) - 1}{\alpha}, \right. \\ &\quad \left. l \frac{\cos(\alpha\pi) - 1}{\alpha}, li \frac{\cos(\alpha\pi) - 1}{\alpha}, lj \frac{\cos(\alpha\pi) - 1}{\alpha}, lk \frac{\cos(\alpha\pi) - 1}{\alpha} \right). \end{aligned}$$

3.1. Convolution Transformation

In general, the convolution is defined as a product between two functions (f and g) that produces a third function, which is typically seen as a modified version of the original functions, then we define the convolution product between two functions as follows.

Let f and g be completely integrable functions, we define the convolution product as the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(u)g(x-u)du. \quad (10)$$

Theorem 2. If $F_{\mathbb{O}}(\alpha)$ and $G_{\mathbb{O}}(\alpha)$ are the Octonionic Fourier Transform of $f(x)$ and $g(x)$, respectively, then the convolution transform of each of the octonionic units is equal to the product of these corresponding transforms, that is,

$$\mathfrak{T}_{\delta}\{f * g\} = F_{\delta}(\alpha)G_{\delta}(\alpha), \quad (11)$$

where $\delta \in \{i, j, k, l, li, lj, lk\}$ and $F_{\delta}(\alpha) = \mathfrak{T}_{\delta}\{f(x)\}$ e $G_{\delta}(\alpha) = \mathfrak{T}_{\delta}\{g(x)\}$.

Proof. We will demonstrate for $\delta = i$. So we have to:

$$\mathfrak{T}_i\{f * g\} = \int_{-\infty}^{\infty} (f * g)e^{-i\alpha x} dx = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-u)g(u)du \right] e^{-i\alpha x} dx.$$

Considering $e^{-i\alpha x} = e^{-i\alpha u}e^{-i\alpha(x-u)}$, obtain

$$\mathfrak{T}_i\{f * g\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-u)g(u)du \right] e^{-i\alpha u}e^{-i\alpha(x-u)} dx.$$

By changing the order of integration, we get:

$$\mathfrak{T}_i\{f * g\} = \int_{-\infty}^{\infty} g(u) \left[\int_{-\infty}^{\infty} f(x-u)e^{-i\alpha(x-u)} dx \right] e^{-i\alpha u} du.$$

By the change of variable, we consider

$$x - u = v \Rightarrow x = u + v \Rightarrow dx = dv.$$

It follows that

$$\begin{aligned} \mathfrak{T}_i\{f * g\} &= \int_{-\infty}^{\infty} g(u) \underbrace{\left[\int_{-\infty}^{\infty} f(v)e^{-i\alpha v} dv \right]}_{F_i(\alpha)} e^{-i\alpha u} du, \\ &= F_i(\alpha) \int_{-\infty}^{\infty} g(u)e^{-i\alpha u} du = F_i(\alpha)G_i(\alpha). \end{aligned}$$

In order to determine the corresponding transform of the other units j, k, l, li, lj, lk is used in the same way. $\square$

Let u, v in R^n , then we define the element-by-element operation $\otimes$ as follows:

$$u \otimes v = (u_1, \dots, u_n) \otimes (v_1, \dots, v_n) = (u_1v_1, \dots, u_nv_n).$$

Theorem 3. If $F_{\mathbb{O}}(\alpha)$ and $G_{\mathbb{O}}(\alpha)$ are the octonions Fourier transforms of $f(x)$ e $g(x)$, respectively, is the Fourier transform of the convolution of $f * g$ corresponding to the product $\otimes$ of these corresponding transforms, that is,

$$\mathfrak{T}_{\mathbb{O}}\{f * g\} = F_{\mathbb{O}}(\alpha) \otimes G_{\mathbb{O}}(\alpha). \quad (12)$$

Proof. From the definition of the octonionic transform and the convolution we obtain

$$\begin{aligned}
 \mathfrak{T}_{\mathbb{O}}\{f * g\} &= (\mathfrak{T}_i\{f * g\}, \mathfrak{T}_j\{f * g\}, \mathfrak{T}_k\{f * g\}, \mathfrak{T}_l\{f * g\}, \\
 &\quad \mathfrak{T}_{li}\{f * g\}, \mathfrak{T}_{lj}\{f * g\}, \mathfrak{T}_{lk}\{f * g\}), \\
 &= (F_i(\alpha)G_i(\alpha), F_j(\alpha)G_j(\alpha), F_k(\alpha)G_k(\alpha), F_l(\alpha)G_l(\alpha), \\
 &\quad F_{li}(\alpha)G_{li}(\alpha), F_{lj}(\alpha)G_{lj}(\alpha), F_{lk}(\alpha)G_{lk}(\alpha)), \\
 &= (F_i, F_j, F_k, F_l, F_{li}, F_{lj}, F_{lk}) \otimes (G_i, G_j, G_k, G_l, G_{li}, G_{lj}, G_{lk}), \\
 &= F_{\mathbb{O}}(\alpha) \otimes G_{\mathbb{O}}(\alpha).
 \end{aligned}$$

□

Example 4. The Dirac delta (see [8]) can be poorly thought as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

The delta function is a tempered distribution and, therefore, it has a well-defined Fourier transform,

$$\mathfrak{T}_i(\delta(\xi)) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} \delta(x) dx = 1.$$

Thus, calculating

$$\begin{aligned}
 F_{\mathbb{O}}(\alpha) &= \mathfrak{T}_{\mathbb{H}}(\delta(\xi)), \\
 &= (F_i(\alpha), F_j(\alpha), F_k(\alpha), F_l(\alpha), F_{li}(\alpha), F_{lj}(\alpha), F_{lk}(\alpha)), \\
 &= (1, 1, 1, 1, 1, 1, 1),
 \end{aligned}$$

consequently we obtain

$$\begin{aligned}
 \mathfrak{T}_{\mathbb{O}}^{-1}(1, 1, 1, 1, 1, 1, 1) &= \frac{1}{14\pi} \int_{-\infty}^{\infty} \langle (1, \dots, 1), (e^{i\alpha x}, \dots, e^{lk\alpha x}) \rangle d\alpha, \\
 &= \delta(\xi).
 \end{aligned}$$

Thus from equation (12) we get

$$\mathfrak{T}_{\mathbb{O}}\{f * \delta\} = \mathfrak{T}_{\mathbb{O}}\{f\},$$

for every f transformable.

4. Conclusion Remarks

In order to generalize the Fourier transform to its octonionic form, first we described Fourier series and we introduced the hypercomplex model by considering results from [8, 9, 12], which are able to obtain a formulation for octonionic Fourier transform, which depends on an internal product. Furthermore, since not few models of Theoretical Physics may be analyzed through the geometry and algebra of hypercomplex, it will be our concern to concentrate the next steps in making all possible applications of our results in the context of unified physical theories for higher dimensional space-times.

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