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ON COVERED LEFT IDEALS OF TERNARY SEMIGROUPS

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Abstract: The notion of covered one-sided ideals of a semigroup was introduced by Fabrici in 1981. In this paper, we introduce covered left ideals and covered right ideals of a ternary semigroup. We study some results of a ternary semigroup containing covered left ideals and give the conditions for every proper left ideal of a ternary semigroup to be a covered left ideal. The results of a ternary semigroup containing covered right ideals can be considered dually.

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1. Introduction and preliminaries

A ternary semigroup is a particular case of n-ary semigroup introduced by Kasner [5], i.e. it is a non-empty set T with an operation $T \times T \times T \to T$, written as $(a, b, c) \to [abc]$, such that $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in T$. Every semigroup can be consider as a ternary semigroup. The notion of ternary semigroups was known for the first time by Banach (cf. [9]) who showed (by an example) that a ternary semigroup does not necessarily reduce to an ordinary semigroup.

Example 1. (Banach's Example) Let $T = \{-i, 0, i\}$. It is easy to see that T is a ternary semigroup under multiplication over complex numbers. Moreover, T is not a binary semigroup under multiplication over complex numbers.

Let A, B and C be non-empty subsets of a ternary semigroup T. A product $[ABC]$ is defined by

$$[ABC] := \{[abc] \mid a \in A, b \in B, c \in C\}.$$

If $A = \{a\}$, then $[\{a\}BC]$ is simply written as $[aBC]$.

In 1965, Sioson [13] studied a ternary semigroup with special reference to ideals and radicals. Now we give the definition of left and right ideal of a ternary semigroup. Let T be a ternary semigroup. A non-empty subset A of T is called a *left* (resp. *right*) *ideal* of T if $[TTA] \subseteq A$ (resp. $[ATT] \subseteq A$). A left ideal A of T is said to be *proper* if $A \neq T$. A proper left ideal A of T is said to be *maximal* if for any left ideal B of T such that $A \subseteq B \subseteq T$, then $A = B$ or $B = T$. If T has no proper left ideal, then it is *left simple*. Ideal theory play an important role in advance studies and uses of algebraic structures. Research of ideal theory of ternary semigroups can bee seen in [1, 3, 11].

It is know that the union of two left (resp. right) ideals of a ternary semigroup T is a left (resp. right) ideal of T, and the intersection of two left (resp. right) ideals of T, if it is non-empty, is a left (resp. right) ideal of T.

Lemma 2. (cf. [10]) *For any non-empty subset A of a ternary semigroup T :*

- *$L(A) = A \cup [TTA]$ is the left ideal generated by A of T ;*
- *$R(A) = A \cup [ATT]$ is the right ideal generated by A of T.*

In a particular case of Lemma 2, for any $a \in T$, we write $L(\{a\})$ (resp.

$R(\{a\}))$ as $L(a)$ (resp. $R(a)$) is the principal left (resp. right) ideal generated by a of T.

Fabrici [4] showed some properties of covered one-sided ideals of semigroups and the relationship between covered one-sided ideals and bases of semigroups. Later, Changphas and Summaprab [2] studied ordered semigroups containing covered one-sided ideals. Recently, in 2019, Khan, Abbasi and Ali [8] studied ordered ternary semigroups containing covered lateral ideals. The purpose of this paper is to introduce covered one-sided ideals including covered left ideals and covered right ideals of a ternary semigroup. The structure of a ternary semigroup containing covered left ideals will be studied. For the results of covered right ideals of a ternary semigroup can be considered similarly.

2. Main results

Firstly, we define covered left ideals of a ternary semigroup.

Definition 3. Let T be a ternary semigroup. A proper left ideal A of T is called a *covered left ideal* (CL-ideal) of T if $A \subseteq [TT(T-A)]$. A covered right ideal of T is defined dually.

Example 4. Let $T = \{0,1,2,3,4,5\}$. Define the ternary operation $[\ ]$ on T by, for all $a,b,c \in T$, $[abc] = (a * b) * c$ where $*$ is the binary operation on T defined by:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

From [12], we have $(T, [\ ])$ is a ternary semigroup. The proper left ideals of T are $A_1 = \{0\}$, $A_2 = \{0,1\}$, $A_3 = \{0,1,2,4\}$ and $A_4 = \{0,1,3,5\}$. We have A_1 and A_2 are covered left ideals of T. Moreover, we have $A_1 = \{0\}$, $A_2 = \{0,1\}$, $A_5 = \{0,1,2,3\}$ and $A_6 = \{0,1,4,5\}$ are proper right ideals of T and we have A_1 and A_2 are covered right ideals of T.

Example 5. Let $T = \{a,b,c,d,e\}$. Define the ternary operation $[\ ]$ on T by, for all $x,y,z \in T$, $[xyz] = x * (y * z)$ where $*$ is the binary operation on T

defined by:

$*$	a	b	c	d	e
a	a	a	c	d	a
b	a	b	c	d	a
c	a	a	c	d	a
d	a	a	c	d	a
e	a	a	c	d	e

Then $(T, [\ ])$ is a ternary semigroup (see [7]). We have $A = \{a, c, d\}$ is a proper left and a proper right ideal of T. Moreover, A is covered right ideal of T since $A \subseteq [(T - A)TT] = T$. But A is not covered left ideal of T since $A \nsubseteq [TT(T - A)] = \{a, b, e\}$.

We have the following useful lemma.

Lemma 6. *Let T be a ternary semigroup. If T contains two different proper left ideals A and B such that $A \cup B = T$, then both A and B are not CL-ideals of T.*

Proof. Assume that T contains two different proper left ideals A and B such that $A \cup B = T$. We have $T - A \subseteq B$ and $T - B \subseteq A$. Suppose that A is a CL-ideal of T. Then $A \subseteq [TT(T - A)] \subseteq [TTB] \subseteq B$. Thus, $A \subseteq B$. Since $A \cup B = T$, it implies $T = B$. This is a contradiction. Hence, A is not a CL-ideal of T. Similarly, if B is a CL-ideal of T, then $B \subseteq [TT(T - B)] \subseteq [TTA] \subseteq A$. Since $A \cup B = T$, we obtain $T = A$. This is again a contradiction. Hence, B is not a CL-ideal of T. □

Corollary 7. *If a ternary semigroup T contains more than one maximal left ideal, then all maximal left ideals are not CL-ideals of T.*

Proof. Assume that T contains two maximal different proper left ideals A and B. We know that union of two left ideals is a left ideal. So, we have $A \cup B$ is a left ideal of T and $A \subset A \cup B$. Since A is a maximal left ideal of T, it implies $A \cup B = T$. Thus, by Lemma 6, neither A nor B is a CL-ideal of T. □

Lemma 8. *Let T be a ternary semigroup. If A is a left ideal of T such that $A \subseteq [TTt]$ and $A \neq [TTt]$ for some $t \in T$, then A is a CL-ideal of T.*

Proof. Assume that A is a left ideal of T such that $A \subseteq [TTt]$ and $A \neq [TTt]$

for some $t \in T$. If $t \in A$, then $[TTt] \subseteq [TTA] \subseteq A$. Thus, $[TTt] \subseteq A$ and so $A = [TTt]$. This is a contradiction. Hence, $t \in T - A$. By assumption, we have $A \subseteq [TTt] \subseteq [TT(T - A)]$. This shows that A is a CL-ideal of T. □

Corollary 9. *A ternary semigroup T in which an element t does not belongs to $[TTt]$ contains CL-ideal.*

Proof. Let $A = [TTt]$. We have $[TTA] = [TT[TTt]] = [[TTT]Tt] \subseteq [TTt] = A$. Thus, A is a left ideal of T. If $t \notin A$, then $A = [TTt] \subseteq [TT(T - A)]$. This implies that A is a CL-ideal of T. □

Lemma 10. *Let T be a ternary semigroup. If A and B are CL-ideals of T, then $A \cup B$ is a CL-ideal of T.*

Proof. Assume that A and B are CL-ideals of T. We have $A \cup B$ is a proper left ideal of T. Next, we will show that $A \cup B \subseteq [TT(T - (A \cup B))]$. Since A and B are CL-ideals of T, we have $A \subseteq [TT(T - A)]$ and $B \subseteq [TT(T - B)]$. Let $x \in A \cup B$. If $x \in A$, then there exists $y \in T - A$ such that $x \in [TTy]$. We consider two cases:

Case 1: If $y \in (T - A) - B$, then $x \in [TTy] \subseteq [TT((T - A) - B)] \subseteq [TT(T - (A \cup B))]$.

Case 2: If $y \in (T - A) \cap B$, we have $y \in B$. Then there exists $z \in T - B$ such that $y \in [TTz]$. If $z \in A$, then $y \in [TTz] \subseteq [TTA] \subseteq A$. Thus, $y \in A$. This contradicts to $y \in T - A$. Hence, $z \in T - A$ and so $z \in (T - A) \cap (T - B) = T - (A \cup B)$. Thus,

$$x \in [TTy] \subseteq [TT[TTz]] = [[TTT]Tz] \subseteq [TTz] \subseteq [TT(T - (A \cup B))].$$

From both cases, we obtain $x \in [TT(T - (A \cup B))]$. Thus, $A \subseteq [TT(T - (A \cup B))]$. Similarly, if $x \in B$, we have $x \in [TTw]$ for some $w \in T - B$. If $w \in (T - B) - A$, then

$$x \in [TTw] \subseteq [TT((T - B) - A)] \subseteq [TT(T - (B \cup A))].$$

If $w \in (T - B) \cap A$, we have $w \in A$ and so $w \in [TTt]$ for some $t \in T - A$. If $t \in B$, then $w \in [TTt] \subseteq [TTB] \subseteq B$. So, we obtain $w \in B$ which is a contradiction. Thus, $t \in (T - B) \cap (T - A) = T - (B \cup A)$ and so

$$x \in [TTw] \subseteq [TT[TTt]] = [[TTT]Tt] \subseteq [TTt] \subseteq [TT(T - (B \cup A))].$$

Hence, $B \subseteq [TT(T - (B \cup A))]$. This shows that $A \cup B$ is a CL-ideal of T. □

Lemma 11. *Let T be a ternary semigroup. If A is a left ideal and B is a CL-ideal of T, then $A \cap B$ is a CL-ideal of T, provided $A \cap B \neq \emptyset$.*

Proof. Assume that A is a left ideal and B is a CL-ideal of T such that $A \cap B \neq \emptyset$. We have $A \cap B$ is a proper left ideal of T. To show that $A \cap B$ is a CL-ideal of T. Since B is a CL-ideal of T, we have $B \subseteq [TT(T - B)]$. Thus, $A \cap B \subseteq B \subseteq [TT(T - B)] \subseteq [TT(T - (A \cap B))]$. This implies $A \cap B$ is a CL-ideal of T. □

Corollary 12. *If A and B are CL-ideals of a ternary semigroup T such that $A \cap B \neq \emptyset$, then $A \cap B$ is a CL-ideal of T.*

Proof. Proof of this corollary is similar to the proof of Lemma 11. □

Theorem 13. *Let T be a ternary semigroup. If T is not a left simple such that there is no any two proper left ideals in which there intersection is empty, then T contains at least one CL-ideal.*

Proof. Assume that T is not a left simple such that there is no any two proper left ideals in which there intersection is empty. Then T contains a proper left ideal A. We have $[TT(T - A)]$ is a left ideal of T since $[TT[TT(T - A)]] = [[TTT]T(T - A)] \subseteq [TT(T - A)]$. By assumption, $A \cap [TT(T - A)] \neq \emptyset$. We let $B = A \cap [TT(T - A)]$. Then B is a proper left ideal of T and $B \subseteq A$. Thus, $T - A \subseteq T - B$ and so $B \subseteq [TT(T - A)] \subseteq [TT(T - B)]$. This shows that B is a CL-ideal of T. □

Let T be a ternary semigroup. A proper left ideal L of T is called the *greatest left ideal* of T if it contains every proper left ideal of T. If a ternary semigroup T contains the greatest left ideal, we denote the left ideal by L^*.

Lemma 14. *Let L^* be the greatest left ideal of a ternary semigroup T. If $T = [TTT]$, then L^* is a CL-ideal of T.*

Proof. Let L^* be the greatest left ideal of a ternary semigroup T. By the proof of Theorem 13, $[TT(T - L^*)]$ is a left ideal of T. Since L^* is the greatest left ideal of T and $[TT(T - L^*)]$ is a left ideal of T, we have $[TT(T - L^*)] = T$ or $[TT(T - L^*)] \subseteq L^*$. We consider three cases:

Case 1: If $[TT(T - L^*)] = T$, then $L^* \subseteq [TT(T - L^*)]$. Thus, L^* is a CL-ideal of T.

Case 2: If $[TT(T-L^*)] = L^*$, then L^* is also a CL-ideal of T.

Case 3: If $[TT(T-L^*)] \subset L^*$, then $[TTT] = [TT(T-L^*)] \cup [TTL^*] \subset L^* \cup L^* = L^* \subset T$. Thus, $[TTT] \subset T$. This contradicts to $T = [TTT]$. □

In Example 4, it is observed that not every proper left ideal of a ternary semigroup is a CL-ideal. In the following theorem we shall find conditions for every proper left ideal of a ternary semigroup is a CL-ideal.

Theorem 15. *Let T be a ternary semigroup such that $T = [TTT]$ which satisfies just one of the following two conditions:*

(1) T contains the greatest left ideal L^*.

(2) For any proper left ideal B and for any element $a \in B$ such that $L(a) \subseteq B$, there is $b \in T - B$ such that $L(a) \subseteq L(b)$.

Then every proper left ideal of T is a CL-ideal of T.

Proof. First, assume that T contains the greatest left ideal L^*. Since $T = [TTT]$, then by Lemma 14, L^* is a CL-ideal of T. Let A be a proper left ideal of T. We have $A \subseteq L^*$ and $T - L^* \subseteq T - A$. Since L^* is a CL-ideal of T, then $A \subseteq L^* \subseteq [TT(T-L^*)] \subseteq [TT(T-A)]$. This implies A is a CL-ideal of T.

Secondary, we assume that T satisfies the condition (2). Let B be a proper left ideal of T and $a \in B$. We have $[TTa] \subseteq [TTB] \subseteq B$. Thus, $L(a) = \{a\} \cup [TTa] \subseteq B$. Then there is $b \in T - B$ such that $L(a) \subseteq L(b)$. Since $T = [TTT]$, we have $b \in [TTt]$ for some $t \in T$. If $t \in B$, then $b \in [TTt] \subseteq [TTB] \subseteq B$. Thus, $b \in B$. This contradicts to $b \in T - B$. Hence, $t \in T - B$. So, we have $b \in [TTt] \subseteq [TT(T-B)]$ and

$$[TTb] \subseteq [TT[TT(T-B)] \subseteq [[TTT]T(T-B)] \subseteq [TT(T-B)].$$

It implies $L(b) = \{b\} \cup [TTb] \subseteq [TT(T-B)]$ and hence $a \in L(a) \subseteq L(b) \subseteq [TT(T-B)]$. Thus, $B \subseteq [TT(T-B)]$. This implies that B is a CL-ideal of T. □

Example 16. Let $T = \{0, a, b, c, d, 1\}$. Define the ternary operation [ ] on T by, for all $x, y, z \in T$, $[xyz] = x * (y * z)$ where $*$ is the binary operation on T defined by:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then $(T, [\ ])$ is a ternary semigroup. Clearly, $[TTT] = T$. The proper left ideals of T are $A_1 = \{0\}$, $A_2 = \{0, a\}$, $A_3 = \{0, b\}$, $A_4 = \{0, a, b\}$ and $A_5 = \{0, a, b, c\}$. Obviously, A_5 is the greatest left ideal of T. Thus, by Theorem 15 (1), every proper left ideal of T is a CL-ideal. Moreover, for each proper left ideal A_i $(i = 1, 2, 3, 4, 5)$ and for any element $x \in A_i$ such that we have $1 \in T - A_i$ such that $L(x) \subseteq L(1)$. Thus, by Theorem 15 (2), for each proper left ideal A_i $(i = 1, 2, 3, 4, 5)$ of T is also a CL-ideal.

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