GENERALIZATION OF A
VARiance-Gamma-Driven
INTERest RATE DERIVATIVE

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Abstract

We derive a generalized Vasicek short rate model under a variance gamma Lévy process by applying Itô lemma, and use the derived model to obtain a generalized interest rate derivative motivated by the variance gamma process.

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1. Introduction

The Lévy processes have contributed to better modelling of phenomenon in different fields (Wei [17], Udoye & Ekhaguere [13], Udoye et al [14]). A variance gamma (VG) process is a type of Lévy process that was launched by Madan and Seneta [7] in order to take care of unexpected occurrences which can lead to inadequate modelling of a given phenomenon. The VG process
is acquired by changing the time of an arithmetic Brownian motion using a gamma process. Seneta [10] and Rathgeber [8] highlighted certain aspects of the process. Since its introduction, it has been applied in different fields which include mathematical finance (Bayazit & Nolder [3], Seneta [11], and Udoye et al [12], [15]), and engineering (Salem [9]), etc. Moreover, Hoyyi [6] discussed the process under Monte-Carlo simulation with closed form method for European call option price valuation. Aguilar [1] discussed different pricing tools of the process, while Azmoodeh et al. [2] emphasized its optimal approximation under second Wiener chaos. Furthermore, Bee et al. [4] highlighted likelihood risk estimates for models of the process. Moreover, Fischer [5] discussed update on distribution theory of the process. This work generalizes the work of Udoye and Ekhaguere [13] who derived an extended Vasicek model under a VG process and used the derived expression to obtain an interest rate derivative driven by the VG process.

In what follows, Section 2 considers important definitions and tools needed in deriving our result. Section 3 concerns the results, while Section 4 concludes the work.

2. Mathematical Notion

**Definition 2.1.** The dynamics of a Vasicek model [16] of an interest rate is given by

\[ dr_t = \varpi (\beta - r_t) dt + \sigma dX_t, \]  

(1)

where \( \varpi, \beta \) and \( \sigma \) denotes speediness of mean reversal, long-standing mean rate and volatility of the interest rate, while \( X_t \) denotes a Lévy process.

**Definition 2.2.** The dynamics of an interest rate derivative called zero-coupon bond price \( P = P_t \) is given by

\[ dP = r_t P dt + \sigma P dX_t, \]  

(2)

where \( \sigma \) is the volatility of the interest rate while \( r_t \) is the interest rate at time \( t \).

**Lemma 2.1.** (Itô formula for Lévy processes)

Let \( X = X_t, t \geq 0 \) be an \( n \)-dimensional Lévy process with characteristic triplet \( (\mathbf{b}, \sigma^2, \nu) \) and a function \( f \in C^{1,2} \) being a map \([0, T] \times \mathbb{R}^n \to \mathbb{R} \). Then,
\[ f(t, X_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x_i}(s, X_{s-}) b_i(t) dX^i_s \]

\[ + 0.5 \int_0^t \sum_{1 \leq i, j \leq n} \sigma^2_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) ds + \sum_{0 \leq s \leq t} \left[ f(s, X_{s-}) + \Delta X_s \right] - f(s, X_{s-}) - \sum_{1 \leq i \leq n} \Delta X^i_s \frac{\partial f}{\partial x_i}(s, X_{s-}) \],

where \(\Delta X_s = X_{s+} - X_{s-}\).

3. Results

To obtain our results, let an extended VG process be given by

\[ X_t = \omega \lambda t + \theta [\lambda G_t + \rho t] + \tilde{\sigma} \sqrt{\lambda G(t)} + \rho t Z, \]  

(3)

where \(\omega = \frac{1}{\kappa} \ln(1 - \theta \kappa - \frac{1}{2} \sigma^2 \kappa)\), \(\kappa\) takes care of variance of the gamma process while \(\theta\) and \(\tilde{\sigma}\) denote parameter for skewness and volatility, respectively, of the arithmetic Brownian motion used to obtain the VG process. \(G = G(t)\) and \(Z = Z(t)\) denote a gamma random variable and a Gaussian random variable, respectively. \(\lambda\) and \(\rho\) are deterministic parameters such that \(0 \leq \lambda, \rho \leq 1\).

**Theorem 3.1.** The generalized Vasicek model driven by a VG process is given by

\[ r_t = r_0 e^{-\omega t} + \beta (1 - e^{-\omega t}) + \sigma \left( \frac{\omega \lambda}{\omega} (1 - e^{-\omega t}) \right) \]

\[ + \frac{\theta \rho}{\omega} (1 - e^{-\omega t}) + \theta \lambda \sum_{0 \leq s \leq t} \Delta G(s) e^{-\omega (t-s)} \]

\[ + \tilde{\sigma} \sum_{0 \leq s \leq t} \Delta \sqrt{\lambda G(s) + \rho s e^{-\omega (t-s)}} Z, \]  

(4)

where \(\Delta X_s = X_{s+} - X_{s-}\) where \(\Delta G(s) = G(s_+) - G(s_-)\).

**Proof.** Applying Itô’s lemma on equation (1) and evaluating, it follows that

\[ r_t = r_0 e^{-\omega t} + \beta (1 - e^{-\omega t}) + \sigma \int_0^t e^{-\omega (t-s)} dX_s. \]
From equation (3),
\[
dX_t = \omega \lambda dt + \theta \lambda \Delta G_t + \theta \rho dt + \tilde{\sigma} \Delta \sqrt{\lambda G(t)} + \rho t Z.
\] (5)

Thus,
\[
\int_0^t e^{-\omega(t-s)} dX_s = \omega \lambda \int_0^t e^{-\omega(t-s)} ds + \theta \lambda \sum_{0 \leq s \leq t} \Delta G(s) e^{-\omega(t-s)} + \theta \rho \int_0^t e^{-\omega(t-s)} ds + \tilde{\sigma} \sum_{0 \leq s \leq t} \Delta \sqrt{\lambda G(s)} + \rho t Z.
\]

Hence, the result follows. \(\square\)

**Theorem 3.2.** The generalized zero-coupon bond price driven by a VG process is given by
\[
P(t, T) = \exp \left( - \left( \frac{T_0}{\omega} (e^{-\omega T} - e^{-\omega t}) + \beta [T - t] - \frac{\beta}{\omega} (e^{-\omega T} - e^{-\omega t}) \right) - e^{-\omega t} + \frac{\sigma \omega \lambda}{\omega} [T - t] + \frac{\sigma \omega \lambda}{\omega} (\frac{1}{\omega} (e^{-\omega T} - e^{-\omega t})) \right) + \frac{\sigma \theta \rho}{\omega} [T - t] + \frac{\sigma \theta \rho}{\omega} \left( \frac{1}{\omega} (e^{-\omega T} - e^{-\omega t}) \right) + \sigma \theta \lambda \times \sum_{t \leq u \leq T} \sum_{0 \leq s \leq t} \Delta G(s) e^{-\omega(u-s)} + \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \sum_{0 \leq s \leq t} \Delta \times \sqrt{\lambda G(s)} + \rho s e^{-\omega(u-s)} Z + \sigma \omega \lambda [T - t] + \sigma \theta \rho [T - t] + \sigma \theta \lambda \sum_{t \leq u \leq T} \Delta G(u) + \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \Delta \sqrt{\lambda G(u)} + \rho u Z - \frac{\sigma^2}{2} \sum_{t \leq u \leq T} (\lambda \Delta G(u) + \tilde{\sigma} \Delta \sqrt{\lambda G(u)} + \rho u Z)^2 \right).
\] (6)

**Proof.** From the dynamics of a zero-coupon bond price given by equation (2),
\[
dP = r_t P dt + \sigma P dX_t.
\]
From Itô’s lemma, \( F(t, x) = \ln x, \frac{\partial F}{\partial t} = 0, \frac{\partial F}{\partial x} = \frac{1}{x} \). Thus,

\[
d\ln P = (r_t dt + \sigma dX_t) - \frac{1}{2} \sigma^2 (dX_t)^2
\]

\[
= r_t dt + \sigma dX_t - \frac{1}{2} \sigma^2 (dX_t, dX_t),
\]

where \( dX_t \) is given by equation (5). Moreover,

\[
(dX_t)^2 = (\theta \lambda \Delta G_t + \tilde{\sigma} \Delta \sqrt{\lambda G(t)} + \rho t Z)^2.
\]

This implies that

\[
d\ln P = r_t dt + \sigma (\omega \lambda dt + \theta \lambda \Delta G_t + \theta p dt + \tilde{\sigma} \Delta \sqrt{\lambda G(t)} + \rho t Z)
\]

\[
- \frac{1}{2} \sigma^2 (\theta \lambda \Delta G_t + \tilde{\sigma} \Delta \sqrt{\lambda G(t)} + \rho t Z)^2.
\]

With \( P(T, T) = 1 \), integrating gives

\[
\ln P(t, T) = -\left( \int_t^T r_u du + \sigma \omega \lambda \int_t^T du + \sigma \theta p \int_t^T du
\right.
\]

\[
+ \sigma \theta \lambda \sum_{t \leq u \leq T} \Delta G(u) + \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \Delta \sqrt{\lambda G(u)} + \rho u Z
\]

\[
- \frac{\sigma^2}{2} \sum_{t \leq u \leq T} \left( \theta \lambda \Delta G(u) + \tilde{\sigma} \Delta \sqrt{\lambda G(u)} + \rho u Z \right)^2
\]

\[
= -\left( \frac{-T^0_\omega}{\omega} (e^{-\omega T} - e^{-\omega t}) + \beta [T - t] + \frac{\beta}{\omega} (e^{-\omega T} - e^{-\omega t})
\right.
\]

\[
+ \frac{\sigma \omega \lambda}{\omega} [T - t] + \frac{\sigma \omega \lambda}{\omega} \left( \frac{1}{\omega} (e^{-\omega T} - e^{-\omega t}) \right)
\]

\[
+ \frac{\sigma \theta p}{\omega} (e^{-\omega T} - e^{-\omega t}) + \sigma \theta \lambda \sum_{t \leq u \leq T \leq s \leq T} \Delta G(s)
\]

\[
\times \ e^{-\omega(u-s)} + \sigma \tilde{\sigma} \sum_{t \leq u \leq T \leq s \leq T} \Delta \sqrt{\lambda G(s)} \ + \rho s e^{-\omega(u-s)} Z
\]

\[
+ \sigma \omega \lambda [T - t] + \sigma \theta p [T - t] + \sigma \theta \lambda \sum_{t \leq u \leq T} \Delta G(u)
\]

\[
+ \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \Delta \sqrt{\lambda G(u)} + \rho u Z
\]

\[
- \frac{\sigma^2}{2} \sum_{t \leq u \leq T} \left( \theta \lambda \Delta G(u) + \tilde{\sigma} \Delta \sqrt{\lambda G(u)} + \rho u Z \right)^2.
\]
Thus,

\[
\ln P(t, T) = -\left( -\frac{\theta}{\omega}(e^{-\omega T} - e^{-\omega t}) + \beta[T - t] + \frac{\beta}{\omega}(e^{-\omega T} - e^{-\omega t}) \right) \\
+ \frac{\sigma \omega}{\omega} [T - t] + \frac{\sigma \omega}{\omega} \left( \frac{1}{\omega}(e^{-\omega T} - e^{-\omega t}) \right) \\
+ \frac{\sigma \rho}{\omega} [T - t] + \frac{\sigma \rho}{\omega} \left( \frac{1}{\omega}(e^{-\omega T} - e^{-\omega t}) \right) \right) \\
\times \sum_{t \leq u \leq T} \sum_{0 \leq s \leq t} \Delta G(s)e^{-\omega(u-s)} + \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \sum_{0 \leq s \leq t} \Delta \\
\times \sqrt{\lambda G(s) + \rho s e^{-\omega(u-s)} Z + \sigma \omega [T - t] + \sigma \rho [T - t]} \\
+ \sigma \theta \lambda \sum_{t \leq u \leq T} \Delta G(u) + \sigma \tilde{\sigma} \sum_{t \leq u \leq T} \Delta \sqrt{\lambda G(u) + \rho u Z} \\
- \frac{\sigma^2}{2} \sum_{t \leq u \leq T} \left( \lambda \theta \Delta G(u) + \tilde{\sigma} \Delta \sqrt{\lambda G(u) + \rho u Z} \right)^2.
\]

Hence, the result in equation (6) follows by taking exponential of the sides of the equation. \( \square \)

4. Conclusion

The generalized version of Vasicek model driven by a variance gamma process and its corresponding interest rate derivative have been derived. These provide a wider atmosphere for different phenomenon to be captured in a financial instrument.

References


